Homework Assignments

Dynamical Systems I

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 21: Show that

$$w(t) = \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(\sigma)\,\mathrm{d}\sigma\right)\,\mathrm{d}s$$

is the solution of the initial-value problem

$$\dot{w}(t) = \beta(t)(\alpha(t) + w(t)), \qquad w(0) = 0.$$

Problem 22: Consider the initial-value problem

$$\dot{x} = f(x) := x, \qquad x(0) = x_0 := 1$$

on the time interval $0 \le t \le T$, for fixed $T > 0$. Calculate an approximate solution $x(T)$ analytically

(i) by Picard iteration, i.e. determine the n-th Picard iterate $x^{[n]}(T)$ via

$$x^{[k+1]}(t) := x_0 + \int_0^t f(x^{[k]}(\tau))\,\mathrm{d}\tau, \qquad x^{[0]}(t) \equiv x_0\,;$$

(ii) by implicit Euler scheme, i.e. determine the value x_n after n Euler steps of stepsize $h = T/n$ via

$$x_{k+1} = x_k + hf(x_{k+1}), \qquad x_0 = x_0\,.$$

Compare the approximations $x^{[n]}(T)$ and x_n with the explicit value of $x(T)$.

Problem 23: Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz continuous. Let $\bar{t}(x_0)$ denote the maximal time of existence for the solution $x(t)$ of the initial value problem

$$\dot{x}(t) = f(x(t)) \qquad x(0) = x_0 \,.$$

Define $m := \inf_{x \in \mathbb{R}^N} \bar{t}(x)$ and $M := \sup_{x \in \mathbb{R}^N} \bar{t}(x)$. Which values can m and M attain?

Problem 24: Let f: $\mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz continuous. Let $J(x_0) = (\underline{t}(x_0), \bar{t}(x_0))$ denote the maximal interval of existence for the solution of the initial-value problem

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0.$$

Prove that the map $x_0 \mapsto \bar{t}(x_0) \in (0, \infty]$ is lower semi-continuous.
Reminder: A map g is called lower semi-continuous at x_0 if

$$\begin{array}{ll} \forall \varepsilon > 0 \ \exists \delta > 0 : \ (|x - x_0| < \delta \ \Rightarrow \ g(x) - g(x_0) > -\varepsilon) \,, & \text{in case} \ \ g(x_0) < \infty, \\ \forall \varepsilon > 0 \ \exists \delta > 0 : \ (|x - x_0| < \delta \ \Rightarrow \ g(x) > 1/\varepsilon) \,, & \text{in case} \ \ g(x_0) = \infty. \end{array}$$