Homework Assignments

Dynamical Systems I

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`http://dynamics.mi.fu-berlin.de/lectures/`

due date: Friday, June 22, 2018

Problem 29: On $u = (p, q) \in \mathbb{R}^N \times \mathbb{R}^N$ consider the symplectic form $\omega[u, v] = u^T J v$,

$$J := \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}.$$

A diffeomorphism Φ is *symplectic*, if the linearization $D\Phi(x)$ preserves the form ω, i.e. if

$$\omega[D\Phi(x)u, D\Phi(x)v] = \omega[u, v]$$

holds for all $x, u, v \in \mathbb{R}^{2N}$. Let Φ_t denote the flow of the C^1 vector field $\dot{x} = f(x)$. Recall that the vector field $f = f(p, q)$ is *Hamiltonian*, if $f = J(D_p H, D_q H)^T$ for a scalar C^2 Hamiltonian function $H = H(p, q)$.

Prove that the flow Φ_t is symplectic if, and only if, the vector field f is Hamiltonian.

Extra credit: Consider evolutions Φ_{t,t_0} of nonautonomous vector fields $f(t, x)$, instead.

Problem 30: [Lissajous figures] Let A be a symmetric real (2×2)-matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}.$$

Consider the Hamilton system with Hamiltonian $H(x, \dot{x}) = \frac{1}{2}(\dot{x}^{\mathrm{T}}\dot{x} + x^{\mathrm{T}}Ax)$, i.e.

$$(*) \qquad \ddot{x} = -Ax.$$

(i) Transform $(*)$ into a system of decoupled pendulum equations (ω_1, ω_2 real):

$$(**) \begin{cases} \ddot{y}_1 + \omega_1^2 y_1 &= 0, \\ \ddot{y}_2 + \omega_2^2 y_2 &= 0, \end{cases}$$

(ii) Sketch the solution $(x_1(t),\ x_2(t))$ of $(*)$ for

$$A = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}$$

with initial conditions $x_1 = x_2 = 1$, $\dot{x}_1 = -1$, $\dot{x}_2 = 1$.

Problem 31: Calculate the solutions of the following linear differential equations

(i) $$\dot{x} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

(ii) $$\dot{x} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Problem 32: Show, that the set of diagonalizable hyperbolic real $N \times N$ matrices,

$$\mathcal{H} := \{A \in M(N, \mathbb{R}),\ \mathrm{spec}(A) \cap \mathrm{i}\mathbb{R} = \emptyset,\ A \text{ diagonalizable}\},$$

is open and dense in the set of all real $N \times N$ matrices. What does that tell you about the growth and decay aspects of the solutions of "most" linear flows $\dot{x} = Ax$.