Homework Assignments

Dynamical Systems I

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`http://dynamics.mi.fu-berlin.de/lectures/`

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Problem 33: Let x_* be an equilibrium of a C^1 flow on $\mathbb{R}^N$. Prove or disprove: x_* is stable, if and only if every neighborhood U of x_* contains a positively invariant neighborhood of x_*.

Problem 34: We want to understand the damped linear pendulum

$$\ddot{x} + \nu\dot{x} + \omega^2 x = 0$$

with parameters $\nu, \omega > 0$ and initial conditions $x(0) = 0$, $\dot{x}(0) = 1$.

(i) Determine the explicit solution of the given initial-value problem for all ν, ω. Sketch phase portraits and a diagram of the (ν, ω)-plane of different qualitative behavior.

(ii) How does the phase portrait change at the boundary between different zones in the (ν, ω)-plane, for instance due to a change of the damping? How do you reconcile the discontinuities in the phase portraits of the Jordan normal form with differentiable dependence of the flow on the parameters ν, ω?

Problem 35: Can an unstable equilibrium become "stable upon linearization"? More precisely, consider $\dot{x} = f(x)$ with $f(0) = 0$. Can the origin be unstable for $\dot{x} = f(x)$, but stable for the linearization $\dot{y} = Df(0)y$? Can it become asymptotically stable? Can an asymptotically stable equilibrium become stable, but not asymptotically stable?
Reminder: An equilibrium x_* of a flow is called unstable if it is not stable, i.e. there exists an $\varepsilon > 0$ such that for any $\delta > 0$ there exists an initial condition x_δ and a time $t(\delta) \geq 0$ such that $|x_\delta - x_*| < \delta$ and $|\Phi_{t(\delta)}(x_\delta) - x_*| > \varepsilon$.

Problem 36: Consider the *Liénard equation* $\ddot{x} + f'(x)\dot{x} + x = 0$, i.e.

$$\begin{cases} \dot{x} = y - f(x), \\ \dot{y} = -x, \end{cases}$$

with $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = 0$. Discuss the local stability of the origin for $f'(0) \neq 0$. What can you say about the global behavior of the solutions if $xf(x) > 0$ for $x \neq 0$? What if $xf(x) < 0$ for $x \neq 0$?