Homework Assignments

Dynamical Systems I

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due date: Friday, July 06, 2018

Problem 37: Let $A \subseteq B \subseteq X = \mathbb{R}^N$ be sets and ϕ_t a flow on X. The set A is called *chain-recurrent* with respect to B if for every $y_0 \in A$ and every $\varepsilon > 0$, $T > 0$ there exists a positive number $n \in \mathbb{N}$, a sequence of times $t_0, \ldots, t_{n-1} \geq T$, and points $y_1, \ldots, y_{n-1} \in B$ such that

$$\operatorname{dist}(\phi_{t_i}(y_i), y_{i+1}) < \varepsilon, \qquad i = 0, \ldots, n-1 \quad (\text{mod } n, \text{ i.e. } y_n := y_0).$$

The set A is called *recurrent*, if we can chose chains of length $n = 1$ for all points, i.e. if $y_0 \in \omega(y_0)$ for all $y_0 \in A$.

Prove: For any $x_0 \in X$, the ω-limit $\omega(x_0)$ is chain-recurrent with respect to X, but it is not necessarily recurrent.

Extra credit: Let the trajectory $\phi_t(x_0)$ be bounded. Prove or disprove: The ω-limit $\omega(x_0)$ is chain-recurrent with respect to *itself*.

Problem 38: Consider the system of differential equations

$$\dot{x}_i \;=\; x_i \left((Ax)_i - x^{\mathrm{T}} A x \right), \qquad i = 1, \ldots, n,$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \geq 0$ and

$$\sum_{i=1}^{n} x_i \;=\; 1.$$

Consider the case of the identity matrix, $A = \mathrm{id}$.

(i) Sketch the phase portraits for $n = 2, 3, 4$.

(ii) Describe the set of equilibria and the set of heteroclinic orbits for arbitrary n. In particular determine which equilibria are connected by heteroclinic orbits.

Extra credit: What happens for $\sum_{i=1}^{n} x_i \neq 1$ and/or $x_i < 0$ for some i ?

Hint: Enumerate equilibria x^* by subsets $M(x^*) = \{i | x_i^* \neq 0\} \subseteq \{1, \ldots, n\}$.

Problem 39: Prove or disprove the following statements.

(i) For gradient flows $\dot{x} = -\nabla V$, $V \in C^1(\mathbb{R}^N, \mathbb{R})$ the critical points of V are the only ω and α limit points.

(ii) Let $x = 0$ be an *isolated* equilibrium of a flow on $\mathbb{R}^N$. If there exists a C^2-Lyapunov function V with $\nabla V(0) = 0$ and indefinite Hessian $\nabla^2 V(0)$, then $x = 0$ is unstable.

Problem 40: **(8 points)** Consider the "monostable KPP equation"

$$\partial_t u(t,x) = \partial_{xx} u(t,x) + ku(t,x)(1 - u(t,x)), \text{ for } t > 0,\ x \in \mathbb{R},\ k \in \mathbb{R} \text{ constant.}$$

A solution of a partial differential equation is called a "traveling wave", whenever it satisfies the traveling wave Ansatz

$$u(t,x) = v(x - ct), \text{ where } c \in \mathbb{R} \text{ is a constant.}$$

Traveling wave solutions $v(\xi)$, $\xi := x - ct$ satisfies the following differential equation

$$v'' = -cv' + kv(v - 1).$$

For $c > 2\sqrt{k}$, prove the existence of traveling waves from 1 to 0:

(i) Study the level sets of the Lyapunov function

$$V(v, v') = \frac{(v')^2}{2} + kv^2(\frac{1}{2} - \frac{v}{3}).$$

(ii) Show by linearization at $v = v' = 0$, that the origin is stable and by linearization at $v = 1, v' = 0$ that there is only one solution $w(\xi)$ converging in backwards time to $v = 1, v' = 0$.

(iii) Show that $V(w(\xi), w'(\xi)) < V(1, 0)$. Hint: Use the fact, that the stable manifold is tangent to the eigenvector of the stable eigenvalue.

(iv) Conclude by LaSalle invariance principle.

(v) *Extra credit:* Show that the solution is positive.