Homework Assignments

Dynamical Systems II

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`http://dynamics.mi.fu-berlin.de/lectures/`

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Problem 17: Consider the pendulum equation

$$\ddot{x} + \nabla V(x) \; = \; 0, \qquad x \in \mathbb{R}^N,$$

with potential $V : \mathbb{R}^N \to \mathbb{R}$,

$$V(x) := \frac{1}{2} x^T A x, \; A = A^T \text{ having eigenvalues } \lambda_1 > ... > \lambda_N \text{ different from zero.}$$

Consider the Hamiltonian $H(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2 + V(x)$.
Prove or disprove: Locally, the level set of the equilibrium $(x, \dot{x}) = (0, 0)$ is the union of its stable and unstable manifolds, i.e.

$$W^{\mathrm{u}}_{\mathrm{loc}}(0,0) \cup W^{\mathrm{s}}_{\mathrm{loc}}(0,0) \; = \; \{(x, \dot{x}) \; ; \; H(x, \dot{x}) = H(0,0)\}_{\mathrm{loc}}$$

(i) for one degree of freedom, $N = 1$;

(ii) for more degrees of freedom, $N > 1$.

Problem 18: Consider a C^k-vector field $\dot{x} = f(x)$, $k \geq 1$, $x \in \mathbb{R}^N$ with associated flow Φ_t. Given a hyperbolic periodic solution $x^*(t)$, $x^*(0) = 0$ with minimal period $p > 0$, define

$$\gamma = \{x^*(t) | t \in \mathbb{R}\}.$$

For a Poincaré section S_0 at 0 with associated Poincaré first return map $P_0 : S_0 \to S_0$ we can define the local stable manifold

$$W^s_{loc}(0) = \{y \in S_0 \cap B_\varepsilon(0) | \lim_{n\to\infty} ||P_0^n y|| = 0\}.$$

Prove or disprove:

$$W^s(\gamma) := \{y \in \mathbb{R}^N | \lim_{t\to\infty} \mathrm{d}(\Phi_t y, \gamma) = 0\} = \bigcup_{t\leq 0} \Phi(t) W^s_{loc}(0).$$

Problem 19: Consider the pendulum

$$\ddot{\varphi} + \sin\varphi \;=\; 0.$$

Let

$$W^s_{loc} \;=\; \{(\varphi, \dot{\varphi}) = (\varphi, h(\varphi - \pi))\;;\; \pi - \varepsilon < \varphi < \pi + \varepsilon\}$$

be the local stable manifold at the equilibrium $\varphi = \pi$, $\dot{\varphi} = 0$. Determine the expansion

$$h(\psi) \;=\; \sum_{k=0}^{N} h_k \psi^k + \mathcal{O}(\psi^{N+1})$$

up to order $N = 3$.

Hint: Use the invariance of W^s.

Extra credit: Determine the corresponding expansion for the damped pendulum

$$\ddot{\varphi} + \alpha\dot{\varphi} + \sin\varphi \;=\; 0$$

with $\alpha > 0$.

Problem 20: Consider a C^k-vector field f in $\mathbb{R}^N$, $k \geq 1$ with global flow $\Phi_t(x)$. Let $x = 0$ be a hyperbolic equilibrium.

(i) Prove that $x = 0$ is a hyperbolic fixed point of the iteration of the time-1 map Φ_1.

(ii) Show that the "discrete" stable manifold of $x = 0$ of the time-1 map is invariant under Φ_t, i.e.

$$x \in W^s_d(0) := \{y \in \mathbb{R}^N \;:\; \lim_{n\to\infty} ||\Phi_n(y)|| = 0\} \Rightarrow \Phi_t(x) \in W^s_d(0), \text{ for all } t \in \mathbb{R}.$$