Homework Assignments

Dynamical Systems II

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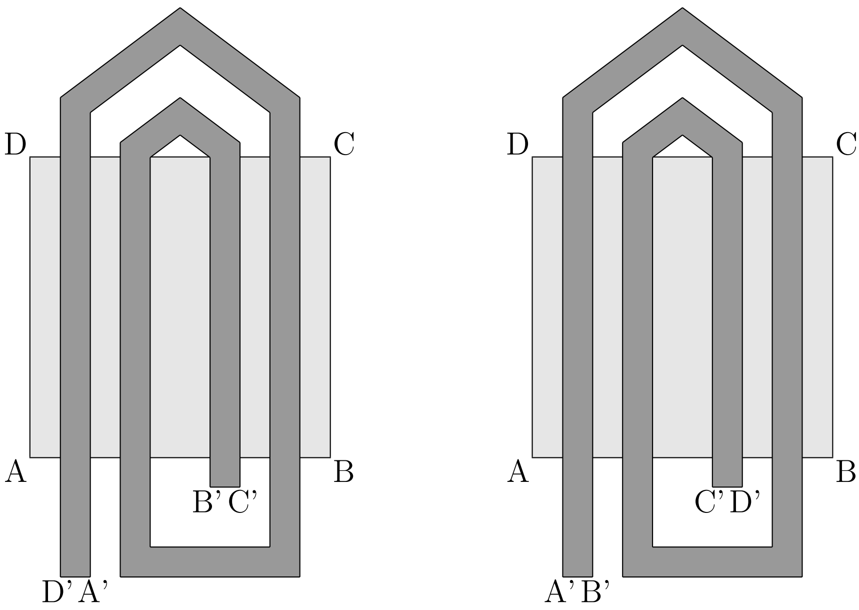
`http://dynamics.mi.fu-berlin.de/lectures/`

due date: Friday, December 7, 2018, 12:00

Problem 25: Which of the following "paper-clip" maps gives rise to shift dynamics? (You can assume the maps to be affine linear, in the regions of intersection.)



Problem 26: A measure of complexity of a map Φ is the *topological entropy* h: Let $N(n)$ be the number of periodic points of Φ with (not necessarily minimal) period n. Then the entropy is defined as

$$h := \limsup_{n\to\infty} \frac{\log N(n)}{n}.$$

Calculate the entropy h of the shift on m symbols. Prove that every iteration Φ containing a shift (i.e. with an invariant set I such that $\Phi|_I$ is conjugate to a shift on m symbols) has positive topological entropy.

Problem 27: Consider the bouncing ball map described by

$$H(x,y) := \begin{cases} 2x - y - \gamma \cos x, \\ x. \end{cases}$$

Here H is a diffeomorphism of $\mathbb{R}^2$. In class we have derived horseshoe dynamics in the square $Q = [0, 2\pi]^2$, for $\gamma > 6\pi$. Let $I \subset Q$ denote the maximal invariant subset of Q. On I, we have proved that H is conjugate to the Bernoulli shift σ on two symbols, i.e. $H = \tau\sigma\tau^{-1}$. *Involutions* are maps for which the second iterate coincides with identity. Let R denote the linear involution

$$\begin{array}{rccc} R: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (x,y) & \mapsto & (y,x). \end{array}$$

(i) Show that H is *reversible* under R, i.e. $H^{-1} = RHR^{-1}$. Conclude that $H = AR$, for a nonlinear involution A.

(ii) Show $RI = I$, i.e. I is also invariant under the linear involution R.

(iii) Characterize the symbolic coding $s = \tau^{-1}(p)$ of all points $p \in I$ which are R-fixed, i.e. $Rp = p$.

(iv) Characterize the symbolic coding of all H-periodic points p which are R-fixed. How many points q of the H-orbit of p also satisfy $Rq = q$?

(v) *Extra credit (4 points):* If p is R-fixed, then the H-orbit of p, as a set, is invariant under R. Conversely, characterize the symbolic coding of any H-orbit which, as a set, is invariant under R. Also characterize the symbolic coding of any H-periodic orbit which, as a set, is invariant under R.

(vi) *Extra credit (4 points):* solve problems (iii) – (v) for the nonlinear involution A, instead of R.

Problem 28: Consider the Hénon-like map

$$\begin{aligned} \bar{x} &= 1 - \alpha x^2 + y, \\ \bar{y} &= \beta x. \end{aligned}$$

Find a horseshoe for parameters $\alpha \gg 1$ and $0 < \beta \ll 1$.

Hint: Choose a small $\varepsilon > 0$ and map the square $Q = [-\varepsilon, \varepsilon]^2$.