Homework Assignments

Dynamical Systems II

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http://dynamics.mi.fu-berlin.de/lectures/

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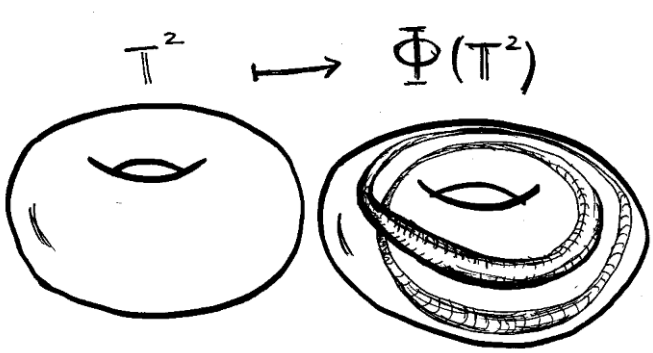
Problem 29: [Smale Solenoid] Realize the following sketch by a continuously differentiable map Φ of the (solid) 2-Torus $\mathbb{T}^2 := \text{disk} \times S^1$ into itself.



(i) Give an example of Φ in suitable coordinates.

(ii) Prove that the attractor $\mathcal{A} := \bigcap_{n=0}^{\infty} \Phi^n(\mathbb{T}^2)$ of Φ is hyperbolic.

Hint: Consider the coordinates (x, y, φ) with $x^2 + y^2 \leq 1$ and $\varphi \in [0, 2\pi)$.

Problem 30: Consider the Smale attractor $\mathcal{A}$ defined in Problem 29. Prove or disprove:

(i) The Smale attractor is connected.

(ii) The Smale attractor is path-connected.

Reminder: A set X is called connected if it cannot be split into two nontrivial open sets, i.e. $U \subset \mathbb{R}^2$ is connected if for all open sets $U, V \subset \mathbb{R}^2$

$$\Big(X \subset U \cup V \text{ and } U \cap V = \emptyset \implies X \subset U \text{ or } X \subset V.\Big)$$

A set X is called path-connected if every two points in X can be connected by a continuous path, i.e.

$$\forall x, y \in X \; \exists f : [0,1] \to X \text{ continuous } : \; f(0) = x \text{ and } f(1) = y.$$

Problem 31: Consider a C^2 differential equation $\dot{x} = f(x)$ with flow Φ_t, where x takes values on a finite dimensional differentiable manifold M. Let $x^*(t)$ be a hyperbolic periodic solution with minimal period $p > 0$ and define the orbit $\gamma \coloneqq \{x^*(t) \mid t \in [0, p]\}$. Prove or disprove:

(i) γ is a hyperbolic set under the flow Φ_t.

(ii) γ is a hyperbolic set under the stroboscope map Φ_p.

Problem 32:

Consider a smooth diffeomorphism $\Phi : M \to M$ defining a hyperbolic splitting on a smooth N-dimensional compact manifold M, $N \geq 2$. Assuming that the expanding direction is non-trivial, prove that the system exhibits sensitivity to initial conditions.

Reminder: A map Φ is sensitive to initial conditions if there exists a $\varepsilon > 0$ such that for all $x_0 \in M$ and $\delta > 0$, there exist y_0 such that $||x_0 - y_0|| < \delta$ and a $n \in \mathbb{N}$ such that $||\Phi^n(x_0) - \Phi^n(y_0)|| > \varepsilon$.