Homework Assignments

Dynamical Systems II

Bernold Fiedler, Alejandro López Nieto
http://dynamics.mi.fu-berlin.de/lectures/

due date: Friday, February 1, 2019, 12:00

The following four exercises comprise a step-by-step analysis of the Takens-Bogdanov bifurcation. Although stated individually (the statement of each problem provides enough information to solve it regardless of whether the previous exercise was solved correctly), they compose a (almost!) complete bifurcation diagram when put together.

References:
J. Guckenheimer and P. Holmes: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, 2003, Sec. 7.3.
V.I. Arnold: Geometrical Methods in the Theory of Ordinary Differential Equations, Springer, 1988.

Problem 45: [Normal form]

Consider a vector field in the normal form

$$\begin{array}{rcl} \dot{x_1} &=& x_2 + a_1 x_1^2 \\ \dot{x_2} &=& a_2 x_1^2 + a_1 x_1 x_2 \end{array} + \mathcal{O}(||(x_1, x_2)||^3), \tag{1}$$

where $a_i \in \mathbb{R}$. Prove that under a suitable change of coordinates equation (1) takes the form of a second order "pendulum equation"

$$\begin{array}{rcl} \dot{y_1} &=& y_2 \\ \dot{y_2} &=& b_1 y_1^2 + b_2 y_1 y_2 \end{array} + \mathcal{O}(||(y_1, y_2)||^3). \text{ Here } b_i \in \mathbb{R}. \tag{2}$$

Problem 46: [Local bifurcations]

Consider the unfolding of equation (2) with respect to the real parameters $\lambda_1, \lambda_2 \in \mathbb{R}$ given by

$$\begin{array}{rcl} \dot{y_1} & = & y_2 \\ \dot{y_2} & = & \lambda_1 + \lambda_2 y_2 + b_1 y_1^2 + b_2 y_1 y_2 \end{array} + \mathcal{O}(||(y_1, y_2)||^3). \tag{3}$$

Rescale the parameters λ_1, λ_2, coordinates y_1, y_2 and time t, possibly also reversing the time direction to convert (3) to the form (4), after truncation to second order.

$$\begin{array}{rcl} \dot{z_1} & = & z_2 \\ \dot{z_2} & = & \mu_1 + \mu_2 z_2 + z_1^2 + z_1 z_2 \end{array}. \tag{4}$$

Determine the number of equilibria and their curves of saddle-node and Hopf bifurcations in the parameter plane (μ_1, μ_2).

Problem 47: [Hamiltonian approximation]

Apply the following spatio-temporal change of coordinates to the equation (4):

$$z_1(t) = \varepsilon^2 \tilde{z}_1(\varepsilon t),\ z_2(t) = \varepsilon^3 \tilde{z}_2(\varepsilon t),\ \mu_1 = \varepsilon^4 \nu_1,\ \mu_2 = \varepsilon^2 \nu_2,\ \varepsilon \geq 0. \tag{5}$$

Study in detail the phase plane dynamics of the resulting pendulum equation with parameter ν_1 for $\varepsilon = 0$.

Problem 48: [Homoclinic loop bifurcation]

(a) Use center manifold theory to prove that, locally $\mu_2 \neq 0$ and negative μ_1 near zero, the equation (4) possesses a heteroclinic orbit connecting its two equilibria e_1 and e_2.

(b) The bifurcation diagram below summarizes the results obtained so far. The periodic orbit in phase diagram (1) disappears somewhere during the passage from (1) to (2), in the left half plane $\mu_1 < 0$. How can that happen? Draw a more complete bifurcation diagram, and sketch the conjectured flow in each section.