Homework Assignments

# Dynamical Systems II

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`http://dynamics.mi.fu-berlin.de/lectures/`

**due date: Friday, January 11, 2018, 12:00**

Only attempted exercises will be discussed. The exercises are for extra credit, i.e. points you obtain will count towards your "Übungsschein", however we will not use these questions to calculate the threshold necessary to obtain the "Übungsschein".

**Weihnachtsaufgabe 1:** Can a flow possess transverse homoclinic or heteroclinic points to:

(i) Equilibria.

(ii) Periodic orbits.

**Weihnachtsaufgabe 2:** Consider a flow in $\Phi$ in $\mathbb{R}^2$ with a hyperbolic saddle fixed point $p = \Phi(p)$. Let $q$ be a transverse homoclinic point to $p$ under iterations of $\Phi$,define the set of transverse homoclinic points

$$T = \{x \in W^u(p) \cap W^s(p) \mid T_x W^u(p) \oplus T_x W^s(p) = \mathbb{R}^2\}.$$

Prove or disprove:

(i) $\#(\{\Phi^n(q) \mid n \in \mathbb{Z}\} \cap T) < \infty$

(ii) $\Phi(q)$ is also a transverse homoclinic point.

(iii) Let $q,\ \Phi(q) \in T$ and let $\gamma(t)$ be an embedded curve in the unstable manifold such that $\gamma(0) = q$ and $\gamma(1) = \Phi(q)$. Then there exists $t^* \in (0, 1)$ such that $\gamma(t^*) \in T$.

**Weihnachtsaufgabe 3:** [Bounds of numerical errors over infinite time]
Let $\Phi : M \to M$ be a continuous map on a metric space $M$. We call a sequence $(\xi_k)_{k\in\mathbb{N}_0}$ a $\delta$-*pseudo orbit*, if the estimate

$$\mathrm{dist}(\Phi(\xi_k), \xi_{k+1}) \ < \ \delta.$$

holds for all $k \in \mathbb{N}_0$. In other words, $\delta$ measures a "numerical error" for the evolution of $\Phi$. We call a true, i.e. error-free $\Phi$-orbit $(x_k)_{k\in\mathbb{N}} = (\Phi^k(x_0))_{k\in\mathbb{N}}$ in $M$ an $\varepsilon$-*shadow* of the pseudo orbit $(\xi_k)_{k\in\mathbb{N}}$ if the estimate

$$\mathrm{dist}(x_k, \xi_k) \ < \ \varepsilon$$

holds for all $k \in \mathbb{N}_0$. We say that the pair $(M, \Phi)$ has the *shadow property*, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every $\delta$-pseudo orbit possesses an $\varepsilon$-shadow.

(i) Prove: the shift on two symbols has the shadow property.

(ii) Give an interpretation of the shadow property from a numerical viewpoint.

*Extra credit:* Is the shadow unique?

**Weihnachtsaufgabe 4:** Let $\mu > 0$ be fixed. Consider a map $u : [0,1] \to [0,1]$ and the horizontal cone $S^+ = \{\, (\xi, \eta) \ : \ |\eta| \leq \mu|\xi| \,\}$.

(i) Prove that $u$ is a horizontal curve (i.e. $u$ is Lipschitz continuous with Lipschitz constant $\mu$) if, and only if, its graph lies inside every horizontal cone attached to it (i.e. graph $(u) \subset S^+ + (x, u(x))$ for every $x \in [0,1]$).

(ii) Let $u$ be differentiable. Prove that $u$ is a horizontal curve if, and only if, every tangent vector lies in $S^+$.

(iii) Formulate similar cone conditions for maps $v : \mathbb{R}^m \to \mathbb{R}^n$.

**Weihnachtsaufgabe 5:** Consider a diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ and a compact, invariant, hyperbolic set $I$. (In particular there exists a hyperbolic structure, i.e. continuous, invariant line bundles $L_p^{\pm}$, $p \in I$, with uniform rates of contraction/expansion.)

Prove or disprove: the hyperbolic structure on $I$ is unique.

**Weihnachtsaufgabe 6:** [Subshift of finite type] Let $\Sigma_N = \{0, 1, \ldots, N-1\}^{\mathbb{Z}}$ be the set of 2-sided sequences on $N$ symbols and $\sigma : \Sigma_N \to \Sigma_N,\ \sigma(x)_k := x_{k-1}$ the shift on $x = (x_k)_{k\in\mathbb{Z}} \in \Sigma_N$.

Consider a matrix $A = (a_{k\ell})_{0\le k,\ell\le N-1} \in \{0,1\}^{N\times N}$ with entries 0 or 1. Let every row and every column of A contain at least one nonzero entry. Define the set

$$\Sigma_A := \left\{ x = (x_k)_{k\in\mathbb{Z}} \in \Sigma_N \;\middle|\; a_{x_k x_{k+1}} = 1 \text{ for all } k \in \mathbb{Z} \right\}$$

of sequences respecting the transfer matrix $A$. The transfer matrix $A$ determines valid successors $x_{k+1}$ of elements $x_k$ in sequences $x \in \Sigma_A$.

Note that $\Sigma_A$ is nonempty, and invariant under the shift $\sigma$. The shift $\sigma$ on the set $\Sigma_A$ is also called a subshift of finite type.

(i) Let
$$\Sigma_{A,n,\alpha,\beta} := \{ (x_0, x_1, \ldots, x_n) \mid x \in \Sigma_A,\ x_0 = \alpha,\ x_n = \beta \}$$
be the set of finite sequences of length $n+1$ which start with $\alpha$ and end with $\beta$. Prove that the number of elements of $\Sigma_{A,n,\alpha,\beta}$ is given by the corresponding entry of $A^n$, i.e.
$$|\Sigma_{A,n,\alpha,\beta}| = (A^n)_{\alpha\beta}.$$

(ii) Prove that the topological entropy is determined by the largest eigenvalue of $A$, i.e.
$$h_{\text{top}} = \log |\lambda_{\max}(A)|.$$

*Reminder:* We defined the *topological entropy* $h_{top}$ in Problem 26 as follows: Let $N(n)$ be the number of periodic points of $\sigma$ with (not necessarily minimal) period $n$. Then $h_{top}$ is defined as

$$h_{\text{top}} := \limsup_{n\to\infty} \frac{\log N(n)}{n}.$$

**Weihnachtsaufgabe 7:** As in **W6** consider the subshift of finite type, i.e. the shift $\sigma$ on the set of sequences $\Sigma_A$ for a given transfer matrix $A = (a_{k\ell})_{0\le k,\ell\le N-1} \in \{0,1\}^{N\times N}$ with entries 0 or 1. Let every row and every column of A contain at least one nonzero entry.

Prove that $\sigma$ possesses a dense orbit in $\Sigma_A$ if, and only if, for every $0 \le k, \ell \le N-1$ there exists a positive integer $n$, such that the matrix entry $(A^n)_{k,\ell}$ is nonzero.

**Weihnachtsaufgabe 8:** Let $\Phi$ be a diffeomorphism of $\mathbb{R}^2$ with a transverse homoclinic orbit to a hyperbolic equilibrium $p$. In class, we proved conjugacy to a shift on two symbols for an iterate $\Phi^n$. Prove that for every $m \in \mathbb{N}$ the shift on $m$ symbols is conjugate to some iterate $\Phi^n$ on a suitable subset of $\mathbb{R}^2$.

**Weihnachtsaufgabe 9:** Can a $C^2$-diffeomorphism $\Phi : S^2 \to S^2$ on the 2-sphere $S^2$ possess a hyperbolic structure.
*Hint:* You can assume the invariant line bundles to be $C^1$.

**Weihnachtsaufgabe 10:** [Horocycles] Consider the POINCARÉ-model of hyperbolic geometry, that is the open upper half plane $\mathcal{H} = \{z = (x, y) \in \mathbb{R}^2 \,:\, y > 0\}$ with the metric tensor

$$g(x,y) = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$$

The geodesics (i.e. locally shortest paths) on $\mathcal{H}$ are known to be the vertical straight lines, $\{z = (x, y) \,:\, x = c,\, y > 0\}$, $c \in \mathbb{R}$, and the (Euclidean) circles with centers on the $x$-axis, $\{z = (x, y) \,:\, (x - c)^2 + y^2 = r^2,\, y > 0\}$, $c \in \mathbb{R}$, $r > 0$.

Consider the geodesic flow $\Phi$ on the unit tangent bundle $T^1\mathcal{H}$. Trajectories of $\Phi$ are given by geodesic curves on $\mathcal{H}$ with attached tangent unit vectors, due to parametrization by arc length.

(i) Show that $(0, y)$ is a unit vector in $T_{(x,y)}\mathcal{H}$, and thus the curve $(z(t), \dot{z}(t)) = ((0, \mathrm{e}^t), (0, \mathrm{e}^t)) \in T^1\mathcal{H}$ is a trajectory of the geodesic flow.

(ii) Consider horizontal lines $W^s(t) := \{((x, \mathrm{e}^t), (0, \mathrm{e}^t)) \in T^1\mathcal{H}; x \in \mathbb{R}\}$. Show that horizontal lines are mapped onto horizontal lines under the geodesic flow. Show that horizontal lines are stable leaves (Blätter/fibers/manifolds), i.e. starting with $(w, \dot{w}) \in W^s(0)$ and considering $z(t)$ as in (i) we have $\mathrm{dist}_{\mathcal{H}}(z(t), w(t)) \xrightarrow{t\to\infty} 0$.

(iii) Prove that the inversion at the Euclidean unit circle

$$\sigma(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right),$$

is an isometry of $\mathcal{H}$, i.e. $g(x, y) = (D\sigma(x, y))^T g(\sigma(x, y))D\sigma(x, y)$. Thus $(\sigma, D\sigma)$ maps trajectories of the geodesic flow onto trajectories. Use this to prove that the unstable leaves $W^u(t)$ of the trajectory (i) are given by circles. Sketch the families $W^u(t)$ and $W^s(t)$.

(iv) Horizontal translations

$$\tau_a(x,y) = (x + a, y)$$

are also isometries of $\mathcal{H}$. Choose $a \neq 0$ and sketch or plot the image of $z(t)$, $W^u(t)$, $W^s(t)$ under the isometry $\sigma \circ \tau_a$.

**Weihnachtsaufgabe 11:** Calculate the center manifold at $(0,0)$ of the vector field

$$\begin{aligned}\dot{x} &= x^2,\\ \dot{y} &= -y,\end{aligned}$$

using:

(i) Separation of variables.

(ii) Power series expansion.

Interpret your result.

**Weihnachtsaufgabe 12:** Consider a smooth vector field $\dot{x} = f(x)$ in $\mathbb{R}^N$, let $0 = f(0)$ be a non-hyperbolic equilibrium. Prove or disprove:

(i) If 0 has an analytic center manifold, it is the only analytic center manifold.

(ii) If 0 has an analytic center manifold, there are no other center manifolds (no particular regularity required).

**Weihnachtsaufgabe 13:** Once upon a time, a talented student took refuge in the countryside, for a few years, from the plague raging at Cambridge. Daydreaming, instead of studying, he watched the snow-white snow flakes gently setting outside on that calm winter day, while pleasant scents of apples from last summer simmering in the kitchen, wafled through his study.

The student returned from his dreams of snow-white to read on, in Latin, about some dusty description of planetary motions:

Planets follow elliptic orbits around the sun, where the sun is located at one of the two foci of the ellipse. Moreover, letting $a$ denote the length of the semi-major axis, the periods $p(a)$ of the orbits satisfy

$$\frac{p(a)^2}{a^3} = \text{const.}$$

In his ODE course, the student had heard about acceleration and speculated about a force law like $\ddot{x} = F(x)$, $x \in \mathbb{R}^2$ (the daydreamer could not find his notes and had forgotten all about mass). Assuming the force $F(x)$ to be directed to the sun, at $x = 0$, and assuming $F(x) = f(|x|)x$ to only depend on distance $|x|$ by a scalar function $f$, help the daydreamer to deduce the law of universal gravitation.