Homework Assignments

Dynamical Systems III-Delay Differential Equations

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 1: Consider the scalar delay differential equation

$$\dot{x}(t) = f(x(t-1)).$$

Here $f \in C^1(\mathbb{R}, \mathbb{R})$. Let $C \coloneqq (C([-1,0], \mathbb{R}), \|\cdot\|)$ where $\|\phi\| \coloneqq \|\phi\|_\infty = \sup_{\theta\in[-1,0]} |\phi(\theta)|$. Using the method of steps, for $t \geq 0$, we can define a family of operators

$$\begin{array}{rcl} T(t) : C & \to & C \\ \phi_0 & \mapsto & \phi_t, \end{array}$$

where, given $\theta \in [-1, 0]$

$$\phi_t(\theta) \coloneqq \begin{cases} \phi_0(t+\theta), & \text{if } -1 \leq t+\theta \leq 0, \\ \phi_n(0) + \int_0^{t+\theta-n} f(\phi_n(\sigma-1))\mathrm{d}\sigma, & \text{if } 0 \leq n < t+\theta \leq n+1,\ n \in \mathbb{Z}. \end{cases}$$

Prove that:

(i) $\{T(t)\}_{t\geq 0}$ is a semigroup, i.e.

- $T(0) = \mathrm{Id}$, and
- if $t, s \geq 0$, it holds that

$$T(t)(T(s)\phi_0) = T(s)(T(t)\phi_0) = T(t+s)\phi_0 = \phi_{t+s}.$$

(ii) Show that the map

$$\begin{array}{rcl} T(\cdot) : [0, +\infty) \times C & \to & C \\ (t, \phi_0) & \mapsto & T(t)\phi_0 = \phi_t, \end{array}$$

is continuous.

Problem 2: In the setting of Problem 1, prove that:

(i) For every $\delta_1 > 0$, there exists $\delta_2 > 0$ such that

$$\|\phi\| \leq \delta_1 \Rightarrow \|T(1)\phi\| \leq \delta_2.$$

(ii) $T(1)$ is compact, i.e. it maps bounded sets to precompact sets in C.

(iii) $T(t)$ is compact for $t \geq 1$.

Hint: Apply Arzelà-Ascoli.

[Extra credit] Ask questions:
Feel free to enclose questions regarding the lecture and the exercises. That way it will be easier to address them during the exercise sessions. Please indicate whether your question should serve as a *basic question* or not.
Asking meaningful questions can earn you up to one extra point!