Homework Assignments

# Dynamical Systems III-Delay Differential Equations

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http://dynamics.mi.fu-berlin.de/lectures/

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**Problem 1:** Given a function $f(t)$, defined for $t \geq 0$, let

$$\mathcal{L}[f(t)](\lambda) \coloneqq \int_0^\infty \mathrm{e}^{-\lambda t} f(t)\mathrm{d}t,$$

denote the Laplace transform of $f$. Throughout this exercise we will consider functions $f$ for which the indicated Laplace transform is well-defined. Derive explicit formulae for:

(i) $\mathcal{L}[f(at)]$ for $a \in \mathbb{R}^+$.

(ii) $\mathcal{L}[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)]$, for $n \in \mathbb{N}$.

(iii) $\mathcal{L}[f(a,t)]$, where

$$f(a,t) \coloneqq \begin{cases} 0, & 0 \leq t < a, \\ 1, & t \geq a. \end{cases}$$

(iv) $\mathcal{L}[g(t)]$, where

$$g(t) \coloneqq \begin{cases} f(t-h), & t \geq h, \\ 0, & 0 \leq t < h. \end{cases}$$

(v) $\mathcal{L}[(f * g)(t)]$, where

$$(f * g)(t) \coloneqq \int_0^t f(\sigma) g(t-\sigma)\mathrm{d}\sigma.$$

Apply the results that you obtained to find an expression for the Laplace transform of the solution of

$$\ddot{x} - 2\dot{x} + x = f(t),$$

where

$$f(t) \coloneqq \begin{cases} \sin(t-1), & t \geq 1, \\ 0, & 0 \leq t < 1. \end{cases}$$

[Extra credit]
(Highly recommended if you have not solved ODEs with Laplace transform before)
Invert the resulting Laplace transform and compare the results to those obtained via variation-of-constants formula.

**Problem 2:** A strongly continuous linear semigroup on a Banach space $X$ is a family of bounded linear operators $T(t) : X \to X$, $t \geq 0$ satisfying that

(i) $T(0) = \mathrm{Id}$.

(ii) $T(t)T(s) = T(t+s)$, $t, s \geq 0$.

(iii) $\lim_{t\to 0^+} ||T(t)x - x|| = 0$ for all $x \in X$.

The corresponding infinitesimal generator is a linear map $A : \mathrm{dom}A \to X$, given by

$$\mathrm{dom}A := \left\{ x \in X \mid \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}, \qquad Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t}.$$

Find the infinitesimal generator $A$ and the domain $\mathrm{dom}A$ of the following semigroups:

(i) The semigroup of solutions of a linear delay equation with $L : C^0([-1,0],\mathbb{R}) \to \mathbb{R}$

$$T^d(t) : C^0([-1,0],\mathbb{R}) \to C^0([-1,0],\mathbb{R}) \qquad \text{solving} \qquad \dot{x}(t) = Lx_t.$$

(ii) The semigroup of solutions of the heat equation

$$T^h(t) : L^2(\mathbb{R}/[0,1],\mathbb{R}) \to L^2(\mathbb{R}/[0,1],\mathbb{R}) \text{ solving}$$

$$\begin{cases} \partial_t u = \partial_{xx} u, & t > 0,\ x \in (0,1), \\ u(t,0) = u(t,1),\ u_x(t,0) = u_x(t,1),\ t \geq 0. \end{cases}$$

(iii) The composition of two strongly continuous linear semigroups $T(t),\ S(\tau) : X \to X$. Assume that $T(t)S(\tau) = S(\tau)T(t)$ for all $t, \tau \geq 0$.

Show that in cases (i) and (ii) the respective domains of the generators are dense sets.

**[Extra credit] Ask questions:**
Feel free to enclose questions regarding the lecture and the exercises. That way it will be easier to address them during the exercise sessions. Please indicate whether your question should serve as a *basic question* or not.
Asking meaningful questions can earn you up to one extra point!