Homework Assignments

Dynamical Systems III-Delay Differential Equations

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http://dynamics.mi.fu-berlin.de/lectures/

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What follows constitutes the basics of the representation theory of linear retarded functional differential equations, we follow [1]. Under this viewpoint both discrete and distributed delays can be studied as the same kind of object. The exercises are a brief introduction to more general linear theory than the one presented in the lectures.

Consider the function space $C = (C^0([-1,0],\mathbb{C}^n), ||\cdot||_\infty)$ and let $\mathcal{L}(C,\mathbb{C}^n)$ denote the dual space of bounded linear maps from C to $\mathbb{C}^n$, equipped with the operator norm. $\mathcal{P}$ denotes the set of finite partitions of the $[-1,0]$ interval, i.e. an element $\sigma \in \mathcal{P}$ of length $m \geq 1$ is a sequence satisfying

$$\sigma = \{\sigma_i\}_{i=0}^m\,,\ -1 = \sigma_0 < \sigma_1 < \cdots < \sigma_m = 0.$$

The diameter of a partition $\sigma \in \mathcal{P}$ is the real value

$$d(\sigma) \coloneqq \max_{i=1,\dots,m} |\sigma_i - \sigma_{i-1}|.$$

For the components of a matrix valued function $\eta = (\eta_{ij}) : [-1,0] \to \mathbb{C}^{n\times n}$ we can define the total variation as

$$V(\eta_{ij}) \coloneqq \sup_{\sigma\in\mathcal{P}} \sum_{i=1}^m |\eta_{ij}(\sigma_i) - \eta_{ij}(\sigma_{i-1})|.$$

η is said to be of bounded variation if $V(\eta_{ij}) < \infty$ for all $i,j = 1,\dots,n$. A function of bounded variation $\eta : [-1,0] \to \mathbb{C}^{n\times n}$ is normalized if

(i) $\eta(-1) = 0$.

(ii) $\lim_{\theta\to a^+} \eta(\theta) = \eta(a)$ for all $a \in (-1,0)$.

A corollary of Riesz representation theorem is that for every element $L \in \mathcal{L}(C,\mathbb{C}^n)$ there exists a unique normalized function of bounded variation $\eta : [-1,0] \to \mathbb{C}^{n\times n}$ such that

$$L\phi = \int_{-1}^0 \mathrm{d}\eta(\theta)\phi(\theta),\ \text{for all } \phi \in C.$$

Here, the object on the right hand side that provides the pairing is the Riemann-Stieltjes integral given by

$$\int_{-1}^0 \mathrm{d}\eta(\theta)\phi(\theta) \coloneqq \lim_{\sigma\in\mathcal{P},\, d(\sigma)\to 0} \sum_{i=1}^m [\eta(\sigma_i) - \eta(\sigma_{i-1})]\phi(\tau_i),\ \text{where } \tau_i \in [\sigma_{i-1},\sigma_i].$$

The limit actually converges to a single complex vector, independently of the choice of partitions σ and points τ_i.

Problem 1: Find the normalized functions of bounded variation $\eta : [-1, 0] \to \mathbb{C}^{n\times n}$ that yield the following linear delay differential equations:

(a) $\dot{x}(t) = x(t-1)$, $\quad n = 1.$

(b) $\begin{cases} \dot{x}(t) &= x(t-1/2) - y(t-1), \\ \dot{y}(t) &= x(t), \end{cases}$ $\quad n = 2.$

(c) $\begin{cases} \dot{x}(t) &= \int_{-1}^{0} y(t+\theta)\mathrm{d}\theta + y(t-1/2), \\ \dot{y}(t) &= \int_{-1/2}^{0} x(t+\theta)\mathrm{d}\theta, \end{cases}$ $\quad n = 2.$

Problem 2: Given a linear delay differential equation $\dot{x} = Lx_t$ where $L \in \mathcal{L}(C, \mathbb{C}^n)$

$$L\phi = \int_{-1}^{0} \mathrm{d}\eta(\theta)\phi(\theta), \text{ for all } \phi \in C.$$

The characteristic function is $\det\Delta(\lambda)$, where

$$\Delta(\lambda) := \lambda I - \int_{-1}^{0} \mathrm{d}\eta(\theta)e^{\lambda\theta}.$$

For equation (c) in Problem 1, compute the characteristic function and prove or disprove that:

(i) The real parts of the roots of the characteristic function are bounded from above, i.e. there exists $\alpha \in \mathbb{R}$ such that if $\det\Delta(\lambda) = 0$ then $\mathrm{Re}(\lambda) < \alpha$.

(ii) Given a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of roots of the characteristic function such that $|\lambda_i| \xrightarrow{i\to\infty} \infty$, then $\mathrm{Re}(\lambda_i) \xrightarrow{i\to\infty} -\infty$.

References:

[1] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel and H.-O. Walther : Delay Equations - Functional-, Complex-, and Nonlinear Analysis, Springer, 1995, Sec. I.1.

[Extra credit] Ask questions:

Feel free to enclose questions regarding the lecture and the exercises. That way it will be easier to address them during the exercise sessions. Please indicate whether your question should serve as a *basic question* or not.

Asking meaningful questions can earn you up to one extra point!