Homework Assignments

Dynamical Systems III-Delay Differential Equations

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 1: [Small delays don't matter]
Consider the delay differential equation

$$\dot{x}(t) = Ax(t) + Bx(t-\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & -4 & -1 \end{pmatrix} x(t) + \begin{pmatrix} -2 & 0 & 0 \\ -2 & -3 & 0 \\ -3 & 4 & 0 \end{pmatrix} x(t-\varepsilon).$$

Show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ the 0 equilibrium of the delay equation is asymptotically stable if, and only if, 0 is asymptotically stable in the ordinary differential equation

$$\dot{y} = (A+B)y.$$

Hint: Use Rouché's theorem.

Problem 2: [Kaplan and Yorke 1974]
Consider $f \in C^1(\mathbb{R}, \mathbb{R})$, such that $f(x) = -f(-x)$, $xf(x) \geq 0$ for all $x \in \mathbb{R}$. Given the delay differential equation

$$\dot{x}(t) = -f(x(t-1)).$$

(i) Show that the delay differential equation does not possess any periodic solutions with period 2.

(ii) Show that if

$$f'(0) > \frac{\pi}{2} \text{ and } \lim_{x \to \infty} \frac{f(x)}{x} = 0,$$

the delay differential equation has a periodic solution x^* with period 4 satisfying $x^*(t-2) = -x^*(t)$.

Solving part (ii) can be split in the following subproblems

(i) A periodic orbit of the kind we wish to find satisfies the planar Hamiltonian ODE

$$\begin{aligned} \dot{u} &= f(v), \\ \dot{v} &= -f(u). \end{aligned}$$

(ii) The planar Hamiltonian ODE has periodic solutions with minimal period 4, this can be proved by observing the period map T implicitly defined as

$$\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \arctan(v(t)/u(t)) \mathrm{d}t = \int_0^T \frac{u\dot{v} - v\dot{u}}{u^2 + v^2} \mathrm{d}t = 2\pi.$$

(iii) Using the reversibility of the system given by $R(u, v) \coloneqq (v, u)$, figure out that the components of a periodic solution with period 4 of the planar ODE actually satisfy

$$v(t) = u(t-1).$$

Thinking about an explicit example could be of help!

[Extra credit] Ask questions:
Feel free to enclose questions regarding the lecture and the exercises. That way it will be easier to address them during the exercise sessions. Please indicate whether your question should serve as a *basic question* or not.
Asking meaningful questions can earn you up to one extra point!