Homework Assignments

Dynamical Systems III-Delay Differential Equations

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 1: [Pyragas stabilization of steady states]
Consider the complex scalar differential equation

$$\dot{z} = (\alpha + i\beta)z.$$

Assuming $\alpha,\ \beta > 0$, it is clear that 0 is an unstable equilibrium. We will now introduce a control term of Pyragas type to obtain

$$\dot{z}(t) = (\alpha + i\beta)z(t) + k(z(t) - z(t - \tau)).$$

Here $k \in \mathbb{R}$, observe that 0 is still an equilibrium of the system. Do the following:

(i) Let $k^* = -\alpha/2$, find the values τ^* for the delay such that the characteristic function admits roots with real part equal zero.

(ii) Given $(k^*, \tau_n) = (-\alpha/2, (2n+1)\pi/\beta)$ for $n \in \mathbb{N}$. Show that for $(\alpha, \beta) = (0.5, \pi)$ there are values $n \in \mathbb{N}$ such that for the control term given by (k^*, τ_n) the real parts of all the eigenvalues are at most 0.

(iii) Given $(\alpha, \beta) = (0.5, \pi)$. Describe the shape of the boundary of stability in the k-τ-plane for one of the regions of successful control.

Problem 2: [Small delays DO matter]
In the lecture we saw how to stabilize discrete rotating waves of the Stuart-Landau ring, i.e. periodic solutions z^* with period p satisfying

$$z_k^*(t) = z_{k-1}^*(t - p/n)$$

and solving the equation

$$\dot{z}_k = (\lambda + i + \gamma |z_k|^2) z_k + a(z_{k-1} - 2z_k + z_{k+1}), \qquad k = 0, \dots, n-1, \ \lambda \in \mathbb{R}, \ a > 0, \ \gamma \in \mathbb{C} \backslash \mathbb{R}_+.$$

More precisely, following [1], for $\tau = p/n$ there exist open regions of control matrices B such that z^* is stable in the controlled system

$$\dot{z}_k = \lambda + i + \gamma |z_k|^2) z_k + a(z_{k-1} - 2z_k + z_{k+1}) + B_k[-z_k(t) + z_{k-1}(t - p/n)].$$

Here B_k denotes the k-th row of B. Under these conditions the maximal coupling constant for which the control works is given by

$$a_{max}(n) \xrightarrow{n \to \infty} \infty.$$

Therefore for the delay we have that $\tau := p/n \xrightarrow{n \to \infty} 0$ and a smaller delay is an improvement in the control!
Justify this observation considering that in Problem 1 in the previous Exercise Sheet we showed that small delays did not bring a significant change in the dynamics of the original ODE.

References:
[1] M. Bosewitz, I. Schneider, *Eliminating restrictions of time-delayed feedback control using equivariance*, Disc. & Cont. Dyn. Sys. (2016).

[Extra credit] Ask questions:
Feel free to enclose questions regarding the lecture and the exercises. That way it will be easier to address them during the exercise sessions. Please indicate whether your question should serve as a *basic question* or not.
Asking meaningful questions can earn you up to one extra point!