Homework Assignments

Dynamical Systems III-Delay Differential Equations

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http://dynamics.mi.fu-berlin.de/lectures/

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This will be the last homework assignment, due to the strange end of semester schedule we included 5 exercises. However, only 3 of them count for the course requirements and the usual one week deadline has been made considerably longer. We will discuss the sheet during the tutorial on Monday July 1st. Best of luck!

Problem 1: [Monotone dynamics]
Consider the scalar delay differential equation

$$\dot{x} = \arctan(x(t-1)).$$

The solution semiflow will be denoted by $T(t)$, $t \geq 0$.

(i) Show that the semiflow $T(t)$ is global, i.e. that the maximum time of existence of any solution is unbounded.

(ii) Let η and ξ be two different solutions, show that $y = \eta - \xi$ solves the linear non-autonomous equation

$$\dot{y}(t) = \left(\int_0^1 \frac{\mathrm{d}\theta}{1 + (\theta\eta(t-1) + (1-\theta)\xi(t-1))^2} \right) y(t-1).$$

(iii) Conclude from (ii) that if $\phi, \psi \in C^0([-1,0])$ satisfy $\phi(t) < \psi(t)$ for all $t \in [-1,0]$, which we will denote $\phi < \psi$, then $T(t)\phi < T(t)\psi$ for all $t \geq 0$.

Problem 2: [A disease model]
The following scalar DDE models the fraction of a population infected by a certain disease, where c is the recovery, b is the infection rate and the delay represents a lag in the propagation, e.g. the time it takes for a host to become contagious.

$$\dot{x}(t) = bx(t-1)(1-x(t)) - cx(t), \; c > b > 0.$$

The solution semiflow will be denoted by $T(t)$.

(i) Show that the region $S := \{\phi \in C^0([-1,0],\mathbb{R}) \mid 0 \leq \phi(\theta) \leq 1 \; \forall \theta \in [-1,0]\}$ is positively invariant, i.e. $T(t)S \subset S$ for all $t \geq 0$.

(ii) Show that 0 attracts asymptotically any solution with initial condition $\phi \in S$. To do this show that

$$V(\phi) := \frac{1}{2c}\phi(0)^2 + \frac{1}{2}\int_{-1}^{0} \phi^2(\theta)\mathrm{d}\theta, \text{ is a Lyapunov function for } \phi \in S.$$

(iii) Give an interpretation for the results obtained in (i) and (ii) and speculate about what could happen by varying $b, \; c > 0$.

Problem 3: [Fiedler et al. 2007]
Given the DDE

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + (x_1^2(t) + x_2^2(t)) \begin{pmatrix} 1 & -\gamma \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$
$$-K \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1(t) - x_1(t-\tau) \\ x_2(t) - x_2(t-\tau) \end{pmatrix}.$$

Here $\lambda < 0$, γ, K and τ are real parameters. For $K = 0$, then

(1) $(\sqrt{-\lambda}\cos((1-\gamma\lambda)t), \sqrt{-\lambda}\sin((1-\gamma\lambda)t))$

is a periodic solution of the system.

(i) Find τ^* such that the (1) is still a periodic solution of the system with $K \neq 0$.

(ii) For $\tau = \tau^*$, solutions of the linearized problem at the target periodic orbit (1) have the form

$$\begin{aligned} y_1(t) &= r(t)[\sqrt{-\lambda}\cos((1-\gamma\lambda)t)] - \varphi(t)[\sqrt{-\lambda}\sin((1-\gamma\lambda)t)] \\ y_2(t) &= r(t)[\sqrt{-\lambda}\sin((1-\gamma\lambda)t)] + \varphi(t)[\sqrt{-\lambda}\cos((1-\gamma\lambda)t)] \end{aligned}$$

where $r(t)$ and $\varphi(t)$ satisfy the real DDE

$$\begin{pmatrix} \dot{r}(t) \\ \dot{\varphi}(t) \end{pmatrix} = \begin{pmatrix} -2\lambda & 0 \\ -2\lambda\gamma & 0 \end{pmatrix} \begin{pmatrix} r(t) \\ \varphi(t) \end{pmatrix}$$
$$-K \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} r(t) - r(t-\tau) \\ \varphi(t) - \varphi(t-\tau) \end{pmatrix}.$$

Show that for $K = 0$, $\lambda\gamma < 1$, the periodic orbit (1) is unstable.

(iii) For $\tau = \tau^*$, compute the Amann-Hooton value that we saw in the lecture

$$-\left(1 + \int_0^\tau \hat{K}_{11}(t)dt\right),$$

where $\hat{K}_{11}(t)$ is the $(1,1)$ entry of the control matrix transformed by the Wronskian $W(t)$ of the uncontrolled system via

$$\hat{K}(t) = KW(t)^{-1} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} W(t).$$

Derive conclusions on instability of the controlled system from the Amann-Hooton value.

Problem 4: [Extra credit]
Consider the scalar delay differential equation

$$\dot{x}(t) = -1/2x(t) + \arctan(x(t-r)).$$

The solution semiflow will be denoted by $T(t)$.

(i) Study the equilibria and their stability with respect to the delay parameter $r > 0$.

(ii) Use the variation-of-constants formula to show that the semiflow $T(t)$ is global in the sense of Problem 1.

Problem 5: [Extra credit]
For n-dimensional linear autonomous delay differential equations with maximal lag $r < \infty$. It is known that small solutions exist if, and only if, the exponential type of the characteristic function is exactly nr. This is equivalent to the injectivity of the solution semiflow. Find, if possible, linear autonomous DDE examples in dimension $n > 1$ that satisfy:

(i) Contains distributed delays and has no small solutions.

(ii) Contains discrete delays and has no small solutions.

(iii) Contains exclusively discrete delays, has no small solutions and the maximal delay is $r > 0$ with phase space $C^0([-R, 0], \mathbb{R}^n)$, $R > r$.

(iv) Contains exclusively distributed delays and has small solutions.

Justify your claims.
Definition:
The exponential type of an entire function $h : \mathbb{C} \to \mathbb{C}$ is

$$E(h) := \limsup_{k\to\infty} \frac{\log(\max_{\varphi\in[0,2\pi]} |h(k\exp(i\varphi))|)}{k}.$$

[Extra credit] Ask questions:
Feel free to enclose questions regarding the lecture and the exercises. That way it will be easier to address them during the exercise sessions. Please indicate whether your question should serve as a *basic question* or not.
Asking meaningful questions can earn you up to one extra point!