Basic Questions of Delay Equations

1. What is a delay differential equation, on the space $C^0([-h,0],\mathbb{C}^n)$? What is the initial value problem for a delay differential equation? In which sense is a delay equation an infinite-dimensional problem?

2. How is the solution to a delay differential equation defined?

3. Give at least one example of a delay equation modelling the "real world". Explain the meaning of the parameters and how the delay arises in the phenomenon that you are modelling.

4. Explain the method of steps for solving a delay equation of the form
$$\dot{x}(t) = \sum_{i=1}^{n} f_i(x(t-\tau_i)),\ \tau_i > 0,\ f_i \in C^0(\mathbb{C}^n, \mathbb{C}^n).$$

5. Solve
$$\dot{x}(t) = x^3(t-1) - x(t-1),$$
with initial condition $x_0 = 1$, using the methods of steps until $t = 2$.

6. Give sufficient conditions under which existence and uniqueness of solutions in delay equations is obtained from existence and uniqueness in ODE. Formulate a theorem and explain.

7. Formulate the most general theorem on existence of solutions of a delay differential equation you know.

8. In the lecture, the proof of existence was done in a similar way to Peano's theorem. Explain the main difference between the ODE and the DDE proof.

9. In what sense do we say that a delay differential equation
$$\dot{x} = f(x_t), \quad f \in C^1(\mathbb{C}^0([-1,0],\mathbb{C}^n),\mathbb{C}^n)$$
has a local solution semiflow? Provide an example of a solution to a delay equation with non-existing backwards continuation.

10. Define the Laplace transform and its inverse. Why is it useful to work with the Laplace transform when solving differential equations?

11. Solve
$$\dot{x}(t) = x^3(t-1) - x(t-1),$$
via Laplace transform.

12. Given a linear scalar delay equation
$$\dot{x}(t) = Ax(t) + Bx(t-1)$$
What is the characteristic function? How do its roots determine the stability of the 0 solution in the linear equation?

13. Consider the delay differential equation
$$\dot{x}(t) = \int_{-2\pi}^{0} \cos(\theta)x(t+\theta)d\theta,$$
what is an exponential Ansatz? Compute the characteristic equation.

14. Cite a theorem which justifies an exponential Ansatz for the scalar equation
$$\dot{x}(t) = Ax(t) + Bx(t-1) + f(t).$$

15. Suppose λ is a root of multiplicity m of the characteristic equation
$$\Delta(\lambda) = \lambda - A - Be^{-\lambda} = 0.$$
Prove: each of the functions $p(t)e^{\lambda t}$ where $p(t)$ is a polynomial of degree $m-1$ is a solution of
$$\dot{x}(t) = Ax(t) + Bx(t-1).$$

16. Consider a scalar, real, delay differential equation of the form
$$\dot{x}(t) = Ax(t) + Bx(t-1).$$
What is its fundamental solution and how does it relate to the characteristic function?

17. Given a non-homogeneous linear scalar delay equation
$$\dot{x}(t) = Ax(t) + Bx(t-1) + f(t)$$
How are its solutions related to the fundamental solution from question 16?

18. Consider a scalar, real, delay differential equation of the form
$$\dot{x}(t) = Ax(t) + Bx(t-1).$$
For the characteristic function $\Delta(\lambda)$ show that:

- The real parts of the roots of $\Delta(\lambda)$ are uniformly bounded from above.
- Given a sequence of roots (λ_n) of $\Delta(\lambda)$, then

$$|\lambda_n| \xrightarrow{n\to\infty} \infty \Rightarrow \mathrm{Re}(\lambda_n) \xrightarrow{n\to\infty} -\infty.$$

- Given $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ there are only finitely many roots of $\Delta(\lambda)$ lying in

$$\{\alpha < \mathrm{Re}(\lambda) < \beta\}.$$

19. What is a small solution of a delay differential equation? How does it relate to the uniqueness of a backwards continuation of a solution?

20. Provide explicit examples for delay equations that:
    - Blow up in finite time.
    - Have solutions that decay to 0 in finite time.

21. When do we say that a function $\eta : [-1, 0] \to \mathbb{C}$ is of normalized bounded variation (NBV)? How are η of NBV related to the linear space of bounded linear functionals $C^0([-1, 0], \mathbb{C})^*$?

22. Why do we say that a delay differential equation is an ill-posed problem in backwards time direction? Is a forward delay equation $\dot{x}(t) = f(x(t + 1))$ also ill-posed backwards in time?

23. In which sense do we say that a delay differential equation regularizes its solutions? Is this also true for a neutral differential equation?

24. Given a periodic solution x^* of a C^1 ordinary differential equation $\dot{x} = f(x)$, $x \in \mathbb{C}^n$. What is a noninvasive control term? Give at least an example involving time delays.

25. Consider a ring of $n = 3$ Stuart-Landau oscillators

$$\dot{z}_k = (\lambda + i + \gamma|z_k|^2)z_k + a(z_{k-1} - 2z_k + z_{k+1}), \qquad z_k \in \mathbb{C},\ \lambda \in \mathbb{R},\ \gamma \in \mathbb{C},\ k (\mathrm{mod}\ n).$$

What is a pony on a merry-go-around solution? Which ones do you expect to arise? How about the cases $n = 4, 5$?

26. Consider a ring of n Stuart-Landau oscillators with nearest neighbor coupling

$$\dot{z}_k = (\lambda + i + \gamma|z_k|^2)z_k + a(z_{k-1} - 2z_k + z_{k+1}), \qquad z_k \in \mathbb{C},\ \lambda \in \mathbb{R},\ \gamma \in \mathbb{C},\ k (\mathrm{mod}\ n).$$

Assume existence of a pony on a merry-go-around solution z^* with period p such that $z_k^*(t) = z_{k-m}^*(t - msp/n)$ for some $m \in \mathbb{N}$ and for all k. We fix a delayed feedback control term

$$\dot{z}_k = (\lambda + i + \gamma|z_k|^2)z_k + a(z_{k-1} - 2z_k + z_{k+1}) + B(z(t)_k - z_{k-m}(t - msp/n)), \qquad B \in \mathbb{C}.$$

For a small, give sufficient conditions on s and m under which there exists B that successfully controls the pony z^*.

27. Formulate the odd-number limitation for delayed feedback control of unstable hyperbolic orbits in non-autonomous ordinary differential equations.

28. Explain why the odd number limitation does not hold in the autonomous case. (The definition of a useful and justified function and a conclusive sketch are enough.)

29. Give the definition of a monotone semiflow on a Banach space X that has a partial order $<$. Give an example of a delay equation whose solution semiflow is monotone.

30. Consider a uniformly bounded, differentiable curve $x : \mathbb{R} \to \mathbb{R}$ satisfying the delay differential equation
$$\dot{x}(t) = -f(x(t), x(t-1)).$$
Here $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$, $\eta f(0, \eta) > 0$ for all $\eta \in \mathbb{R}$ and $\partial_\eta f(\xi, \eta)|_{(\xi,\eta)=(0,0)} > 0$. Define the zero number of $x_t := x(t+\theta)$, $\theta \in [-1, 0]$ and formulate the zero dropping theorem.

31. What is a strongly continuous, linear semigroup on $C^0([-1, 0], \mathbb{C})$? What is its infinitesimal generator?

32. What is the main advantage of the sun-star calculus approach to delay equations?

33. Given a linear delay equation
$$\dot{x}(t) = Ax(t) + Bx(t - \varepsilon),$$
where $A, B \in \mathbb{C}^{n \times n}$. What can be said about the stability of 0, for $\varepsilon > 0$ very small?

34. Given a linear scalar delay equation
$$\dot{x}(t) = Ax(t) + Bx(t - r), \; r \gg 0.$$
What is the pseudocontinuous spectrum? And what is the strongly unstable spectrum?

35. Consider the linear scalar delay equation
$$\dot{x}(t) = Ax(t) + Bx(t - r), \; r \gg 0.$$
Under which conditions do we have stability of a large delay r? Sketch the spectrum in that case. Pay attention to the appropriate rescaling of the axes!

36. What is a neutral differential equation? Is backwards continuation of neutral equations possible? If yes, how?

37. Name two major differences between neutral and delay differential equations.

Disclaimer: We do not guarantee completeness nor correctness of the questions. They should give a good idea of what was covered in the course and what we expect our students to master, however, the questions in the exam could be considerably different. Do not forget to go over the exercises!