Local Bifurcation Theory: ten examples

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SFB 910

Parameter (in-)dependence

Parameter dependence

Question: Dependence of $x(t)$ on parameter λ?

$$\dot{x}(t) = f(\lambda, x(t)) \qquad \text{(bifurcation theory)}$$
$$0 = f(\lambda, x(t)) \qquad \text{(singularity theory)}$$

Example 1: nonzero flow, locally

$$\dot{x} = f(\lambda, x) \neq 0\,; \qquad \text{flow box: no change!}$$

Example 2: hyperbolic equilibrium, locally

$$\dot{x} = f(\lambda, x) = Ax + \ldots\,, \qquad 0 \notin \mathrm{Re}(\mathrm{spec}\, A)\,;$$

[Grobman, Hartman] (1959,1963) no change!

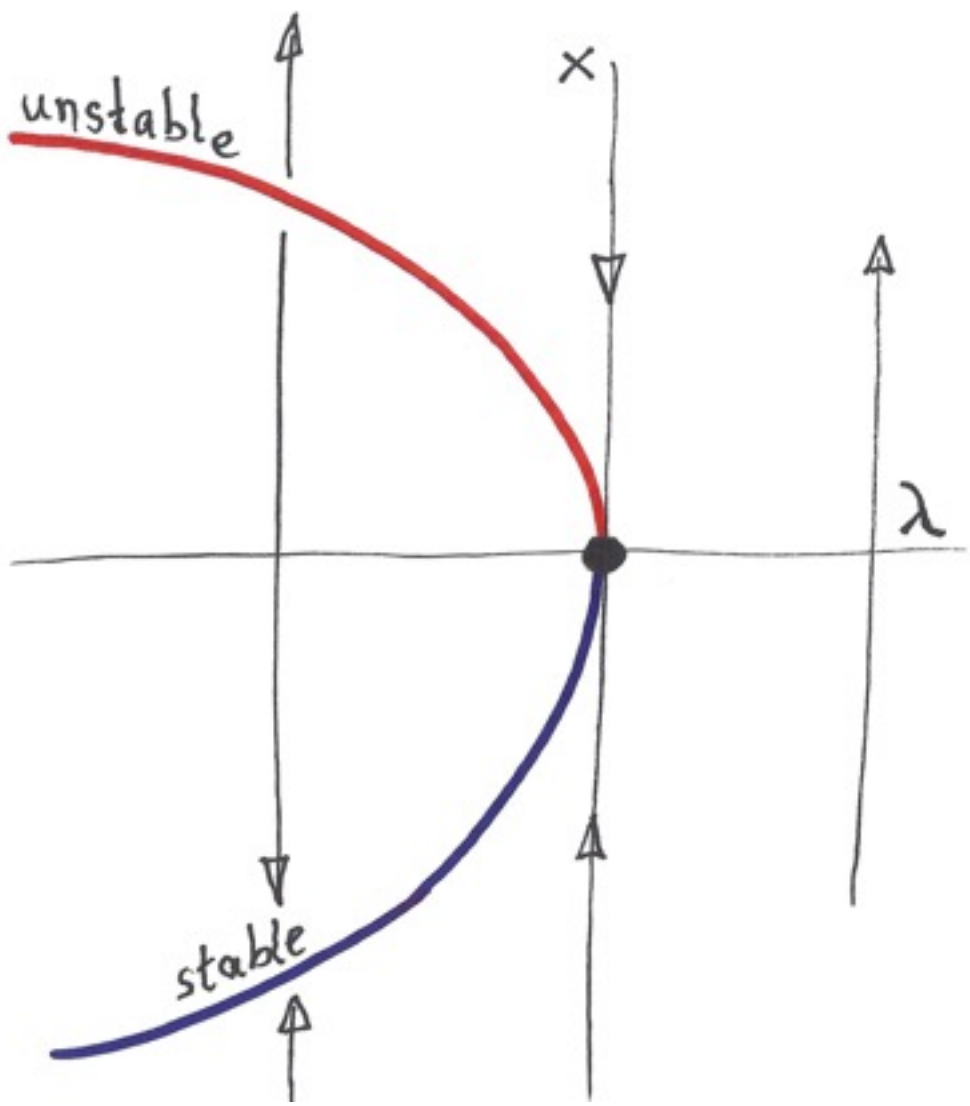
Dimension one ...

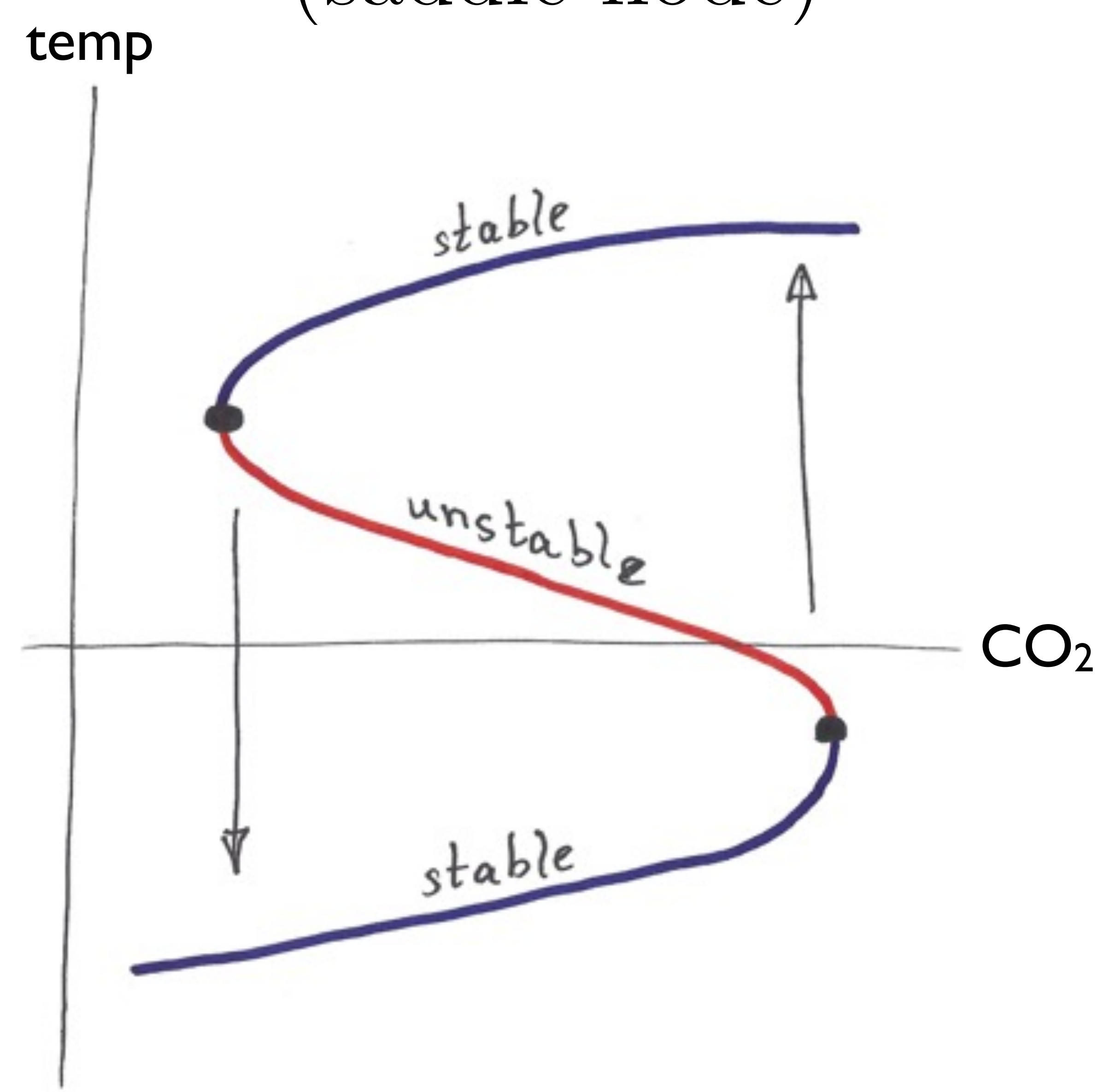
Example 3: stationary bifurcation

Setting: $\dim x = 1, \ \operatorname{spec} A = \{0\}$

$$\dot{x} = \lambda + x^2$$

(saddle-node)



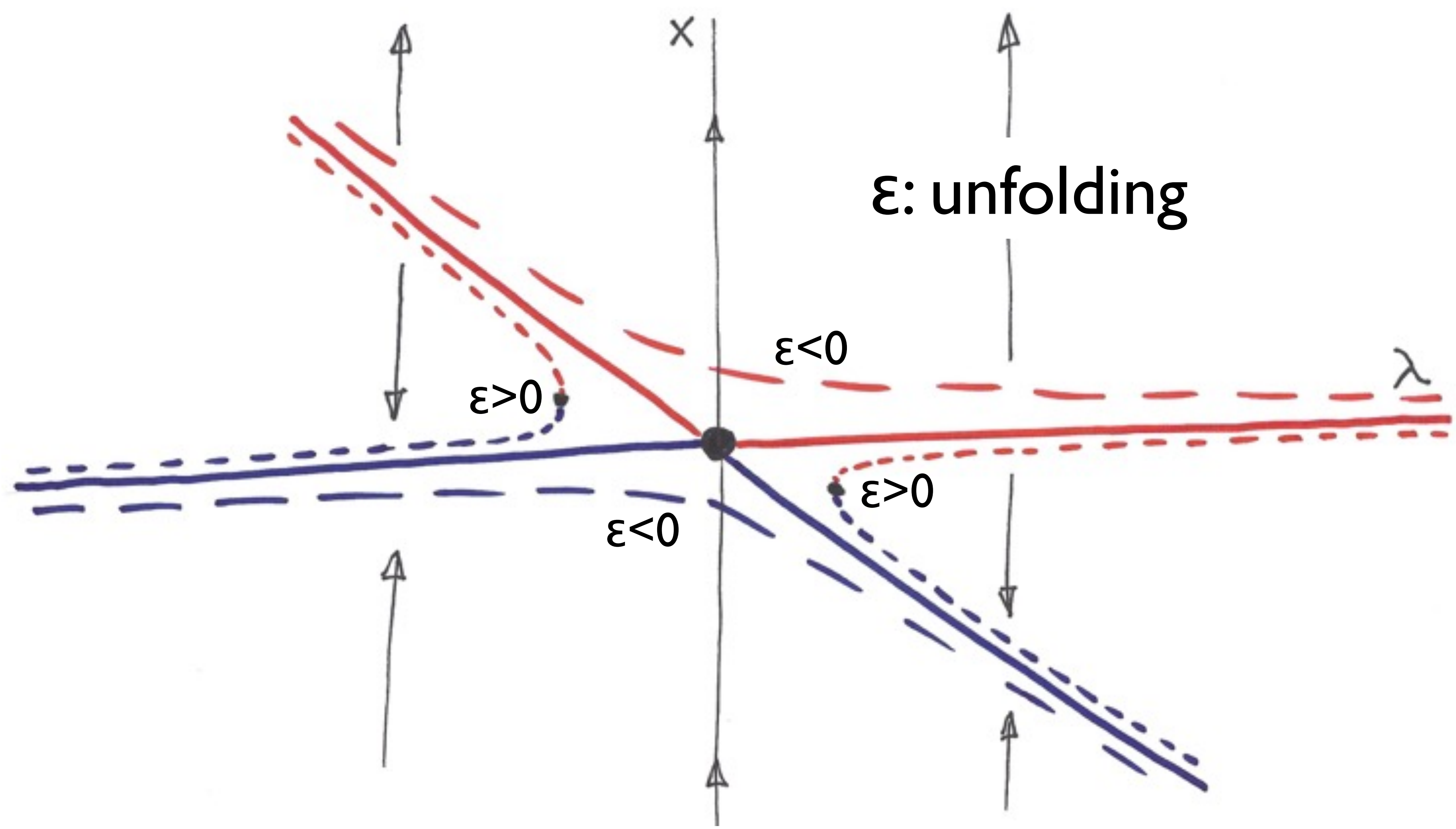


Example 3: stationary bifurcation

Setting: $\dim x = 1, \ \mathrm{spec}\, A = \{0\}$

$$\dot{x} = \lambda + x^2 \qquad \text{(saddle-node)}$$

$$\dot{x} = x(\lambda + x) + \varepsilon \qquad \text{(transcritical)}$$



Crossing of BZ scroll filaments

[Fiedler, Mantel](2000)
[Hauser, Kupitz](2014)

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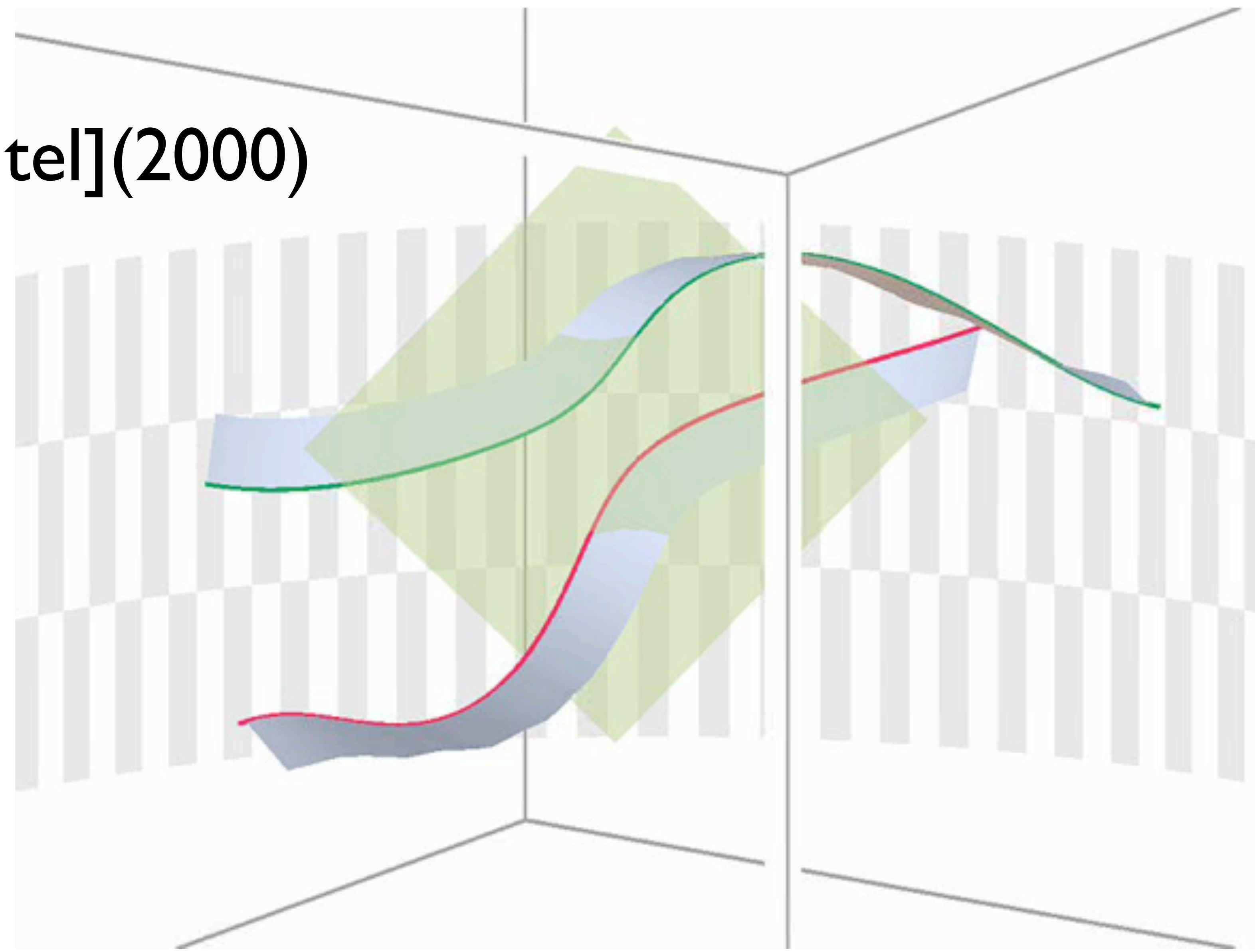

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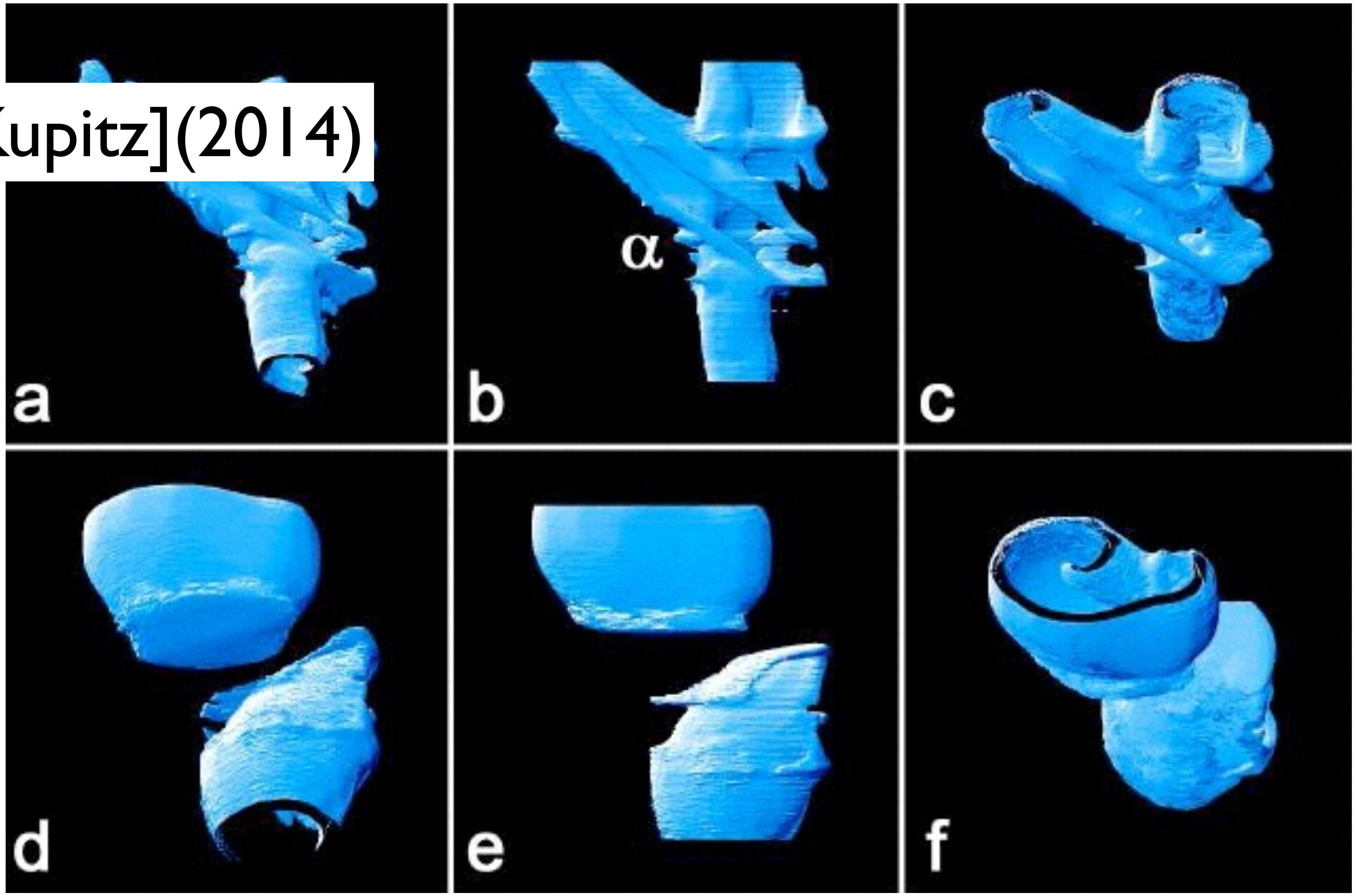


Example 3: stationary bifurcation

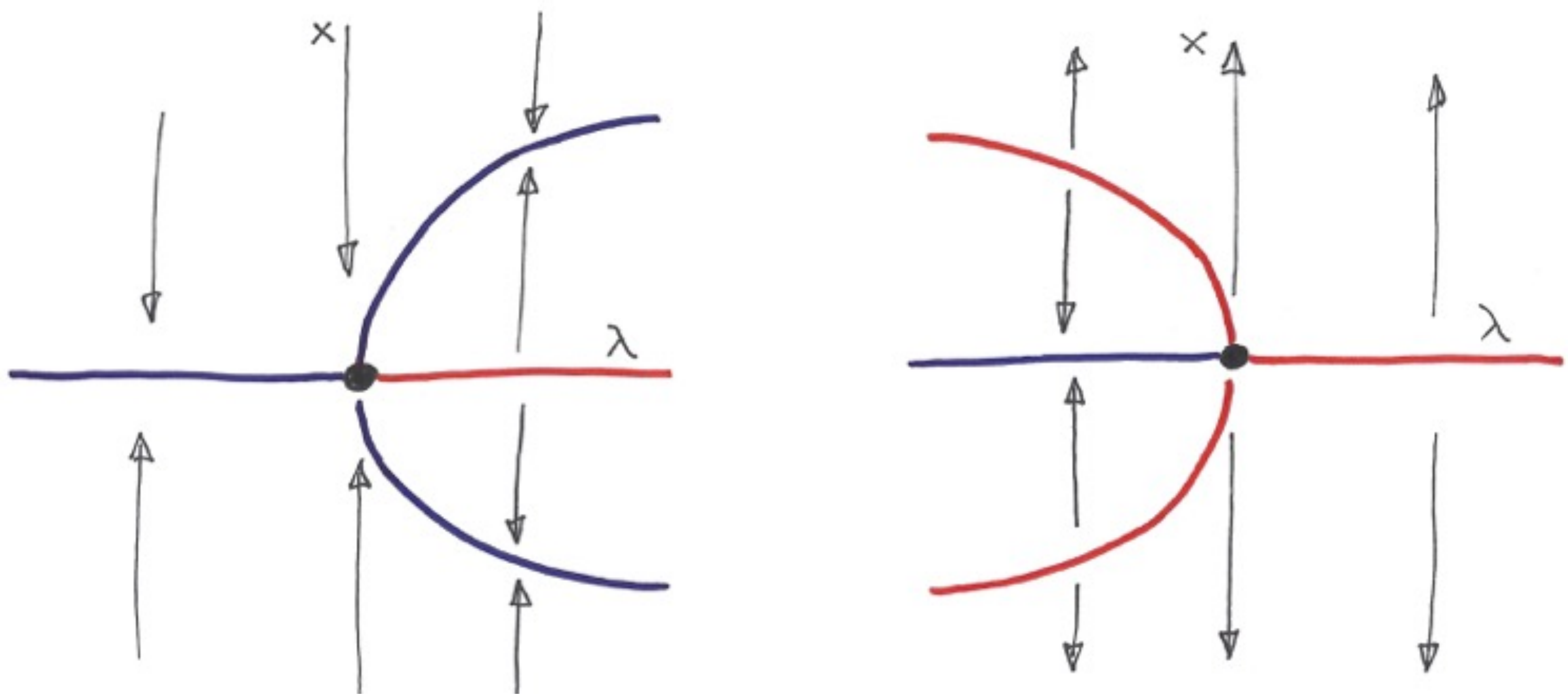
Setting: $\dim x = 1, \; \mathrm{spec}\, A = \{0\}$

$\dot{x} = \lambda + x^2$ (saddle-node)

$\dot{x} = x(\lambda + x) + \varepsilon$ (transcritical)

$\dot{x} = x(\lambda - x^2)$ (pitchfork, supercritical)

$\dot{x} = x(\lambda + x^2)$ (pitchfork, subcritical)



Example 4: period doubling

Setting: iteration, $\dim x = 1, \ \operatorname{spec} A = \{-1\}$

$$x_{n+1} = x_n(\lambda - 1 \pm x_n^2) \qquad \text{(iteration)}$$

$$x_n = 0 \qquad \text{(fixed point)}$$

$$-x_n = x_{n+1} = x_n(\lambda - 1 \pm x_n^2) \qquad \text{(period 2)}$$

$$\Longleftrightarrow \qquad 0 = \lambda \pm x_n^2 \qquad \text{(pitchfork)}$$

Example 4: period doubling

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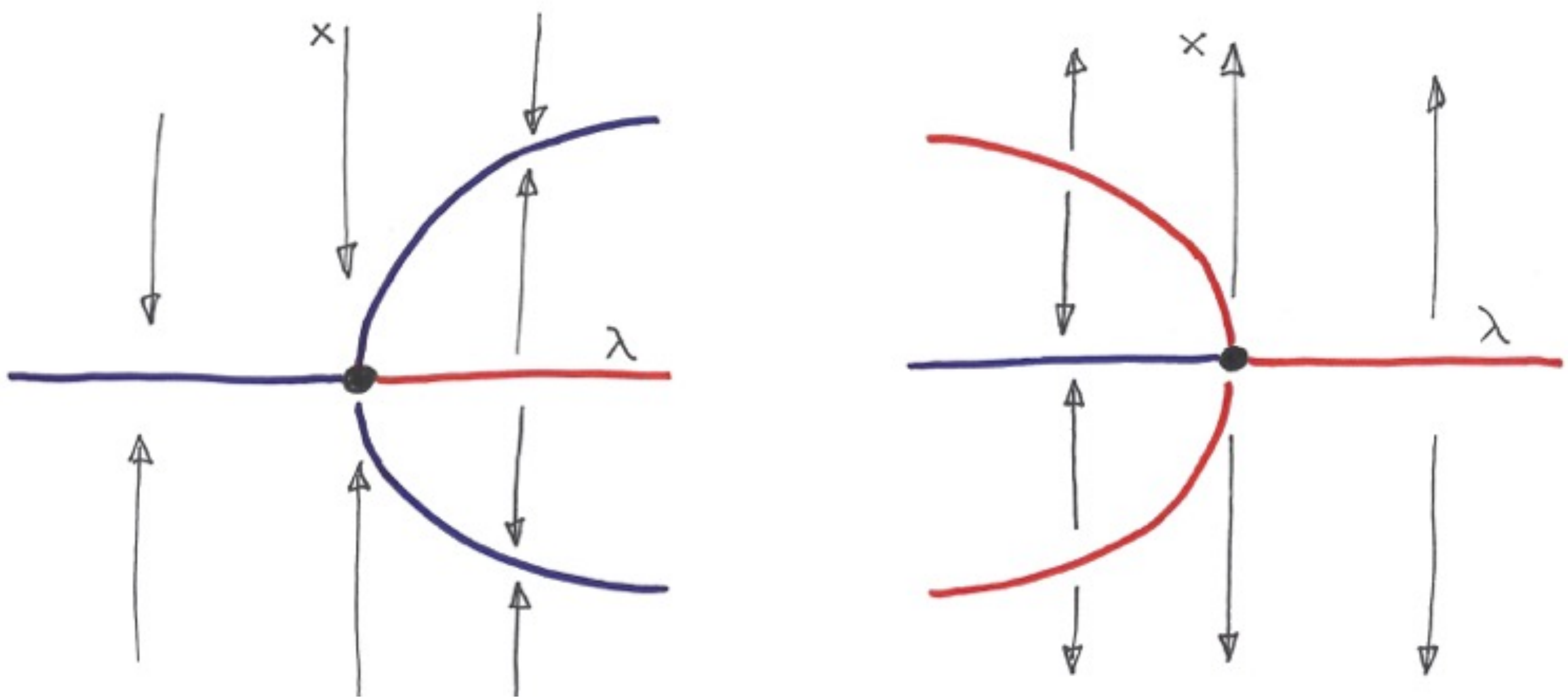

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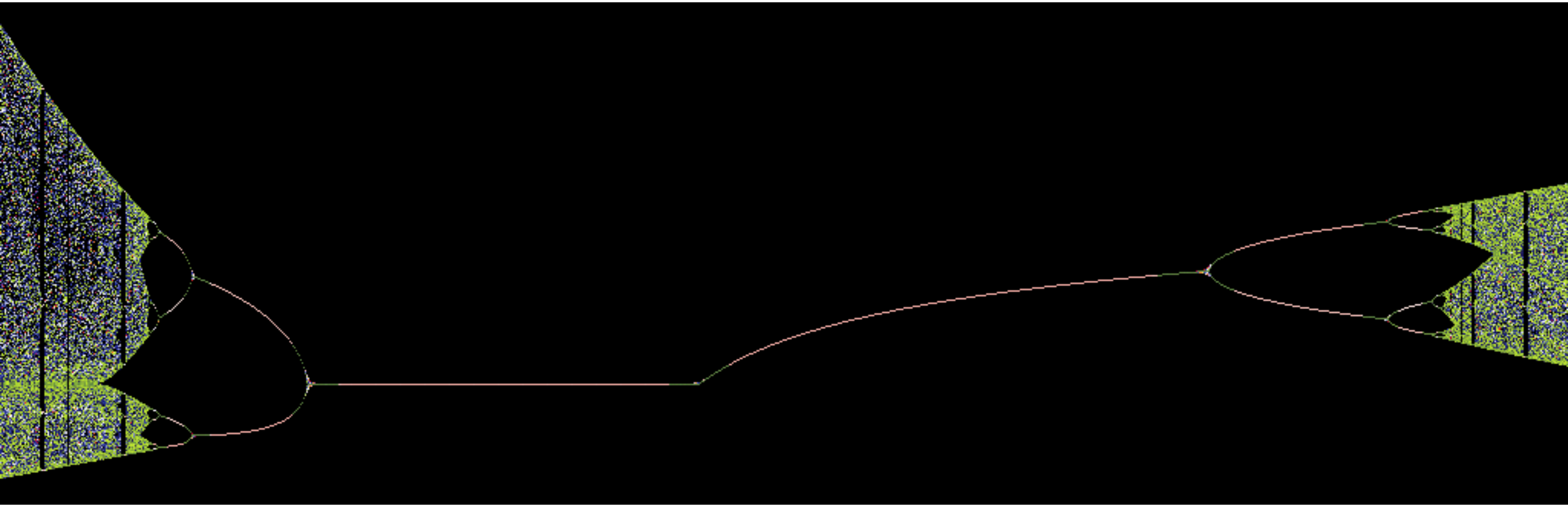


$\iff \qquad 0 = \lambda \pm x_n^2 \qquad$ (pitchfork)

Dimension one … and two

purely imaginary eigenvalues

Henri Poincaré (1899)
Alexander A. Andronov (1933)
Eberhard Hopf (1942)

Example 5: Hopf bifurcation

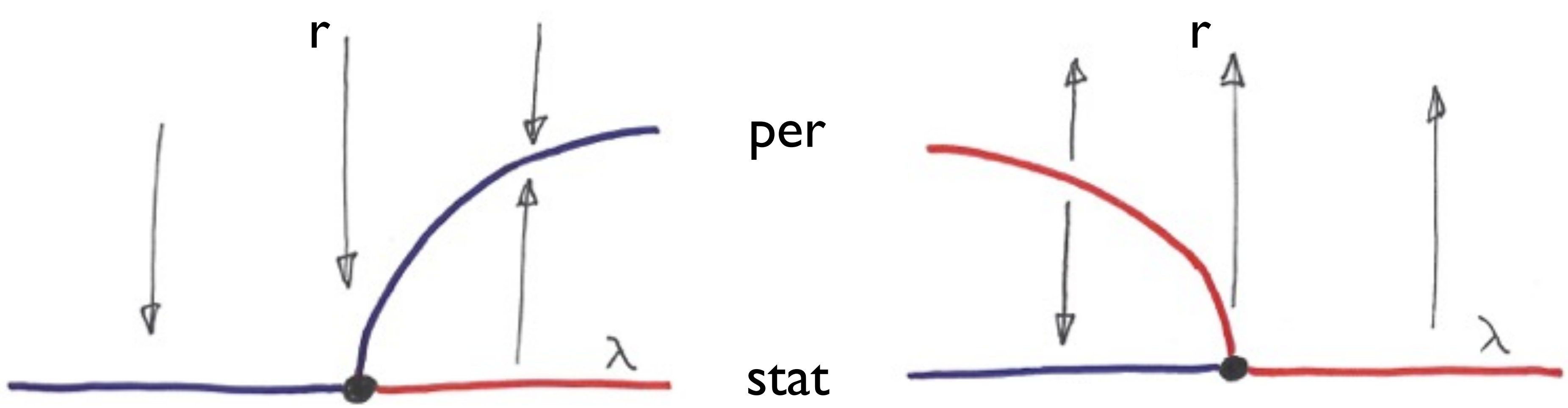
Setting: $\dim x = 2,\ \operatorname{spec} A = \{\pm \mathrm{i}\omega\}$
Stuart-Landau oscillator

$$\dot{z} = z(\lambda + \mathrm{i}\omega + (\pm 1 + \mathrm{i}\gamma)|z|^2)\,, \qquad z = r\exp(\mathrm{i}\varphi)$$

$$\dot{r} = r(\lambda \pm r^2) \qquad \text{(pitchfork)}$$

$$\dot{\varphi} = \omega + \gamma r^2 \qquad \text{(frequency)}$$

Example 5: Hopf bifurcation



Some oscillatory reactions

1828 [Fechner] electrochemistry (single polarity reversal)
1901 [Heathcoate] electrochemical oscillations
1920 [Lotka] NOT predator-prey, 1921 [Bray] experiment
1931 [Volterra] population dynamics
1958 [Belousov] experiment (rejected), 1964 [Zhabotinsky]
1967 [Prigogine et al] Brusselator
1967 [Higgins] 2d-survey
1968 [Sel'kov] glycolysis
1970 [Luss, Lee] nonisothermal PDE catalysis
1971 [Eigen] hypercycle
1974 [Field et al] Oregonator
1983 [Fiedler] PDE catalysis; global Hopf bifurcation
1990 [Dupont, Goldbeter] cellular Ca^{++}
1995 [Imbihl, Ertl] survey PDE catalysis
2003 [MacDonald et al] Citric acid cycle
2009 [Mirsky et al] Mammalian circadian clock

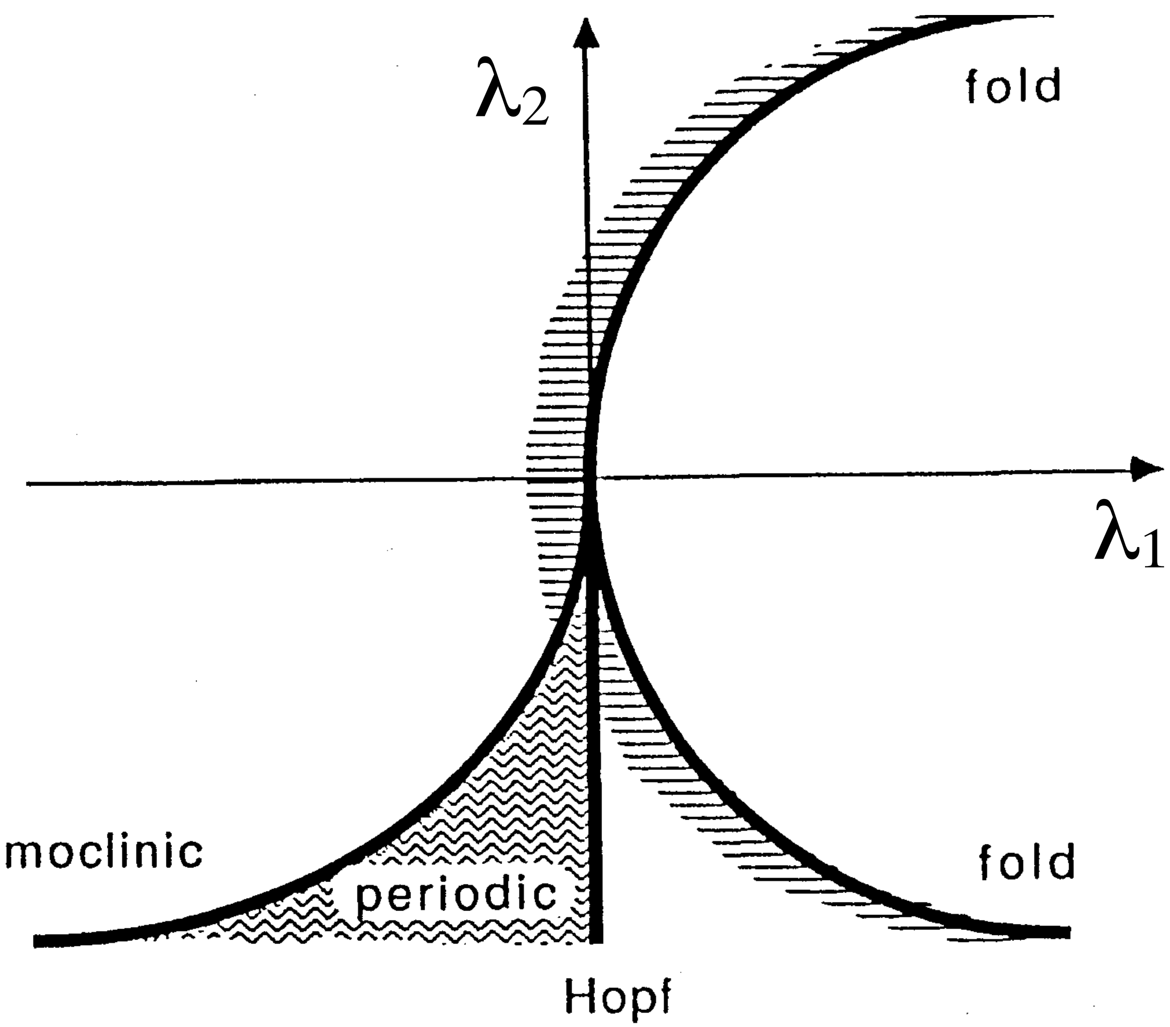
double zero eigenvalue

Vladimir I. Arnold (1971)
Floris Takens (1973)
Rifkat I. Bogdanov (1976)

Example 6: Bogdanov-Takens bifurcation

Setting: $\dim x = 2,\ \operatorname{spec} A = \{0\}$, alg. double

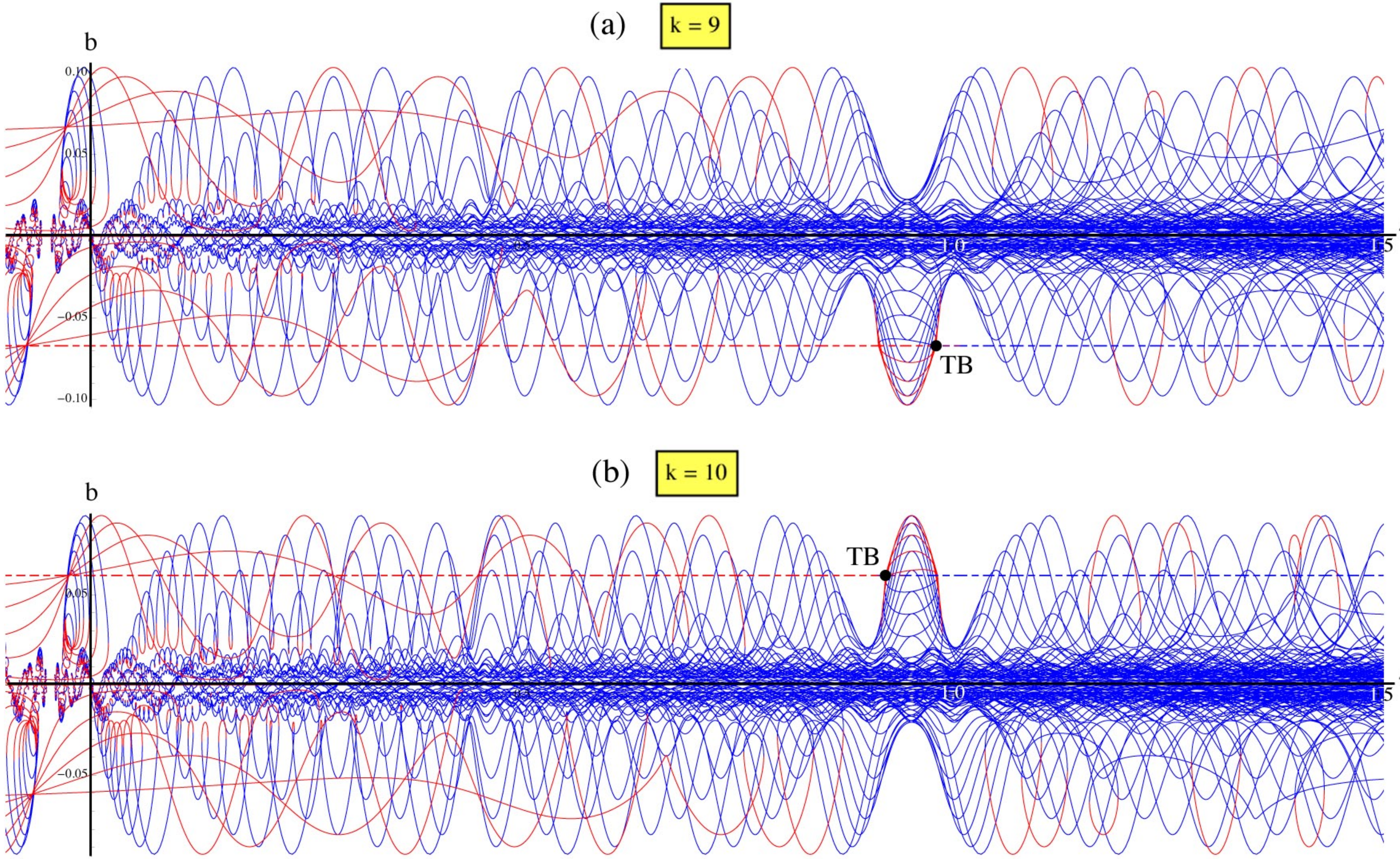
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \lambda_1 + \lambda_2 x_1 + x_1^2 \pm x_1 x_2$$



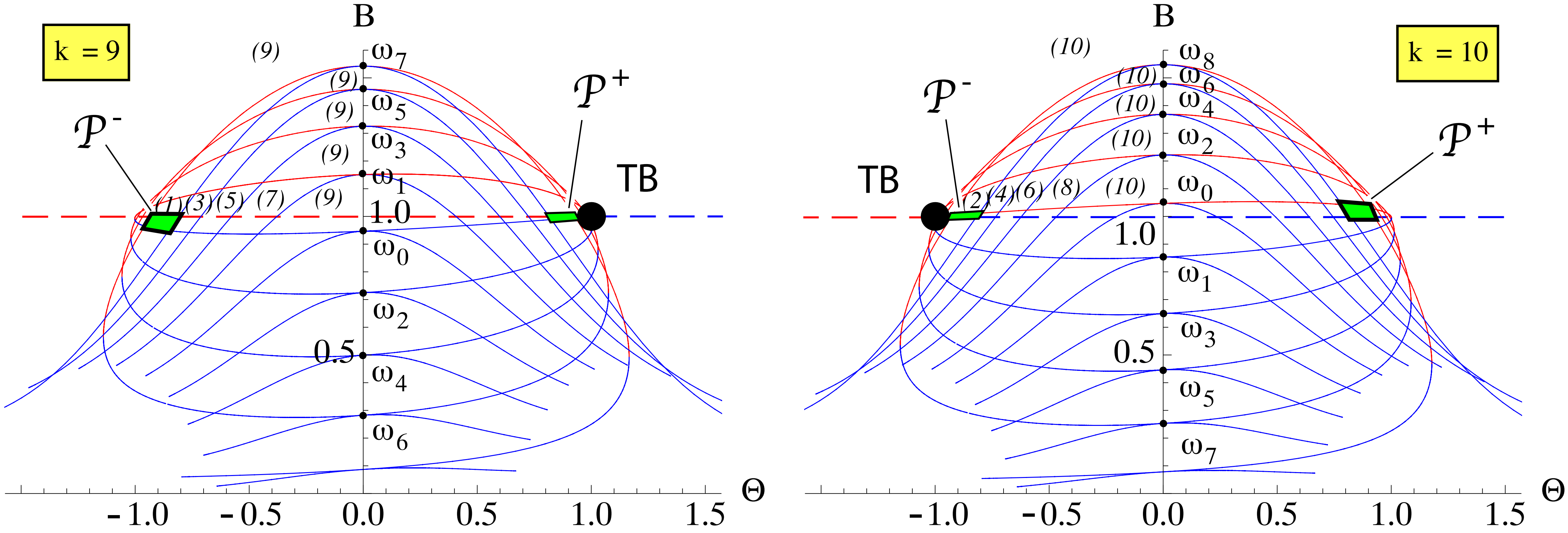
delay controls delay

[Fiedler, Oliva](2015)

Application: Pyragas delays



Application: Pyragas delay



Justifications

Justifications of dimension reduction

Setting: $\dot{x} = Ax + \dots$

- normal form theory $\longrightarrow$ $\exp(A^T t)$ equivariance
- Lyapunov-Schmidt reduction
- center manifold theorem
- Shoshitaishvili theorem:

$$\dot{x}^c = h(\lambda, x^c) \qquad \text{(center part)}$$
$$\dot{x}^h = A^h x^h \qquad \text{(hyperbolic part)}$$

Symmetry-breaking

Pascal Chossat

Martin Golubitsky

Reiner Lauterbach

Ian Stewart, ...

Equivariance and symmetry breaking

Setting: matrix group $g \in G \leq \mathbb{O}(N)$

$$\begin{aligned} \dot{x} &= f(x) \\ f(gx) &= gf(x), \qquad \text{for all } g \text{ and } x, \text{ i.e.} \\ x(t) \text{ solves } &\Leftrightarrow gx(t) \text{ solves} \end{aligned}$$

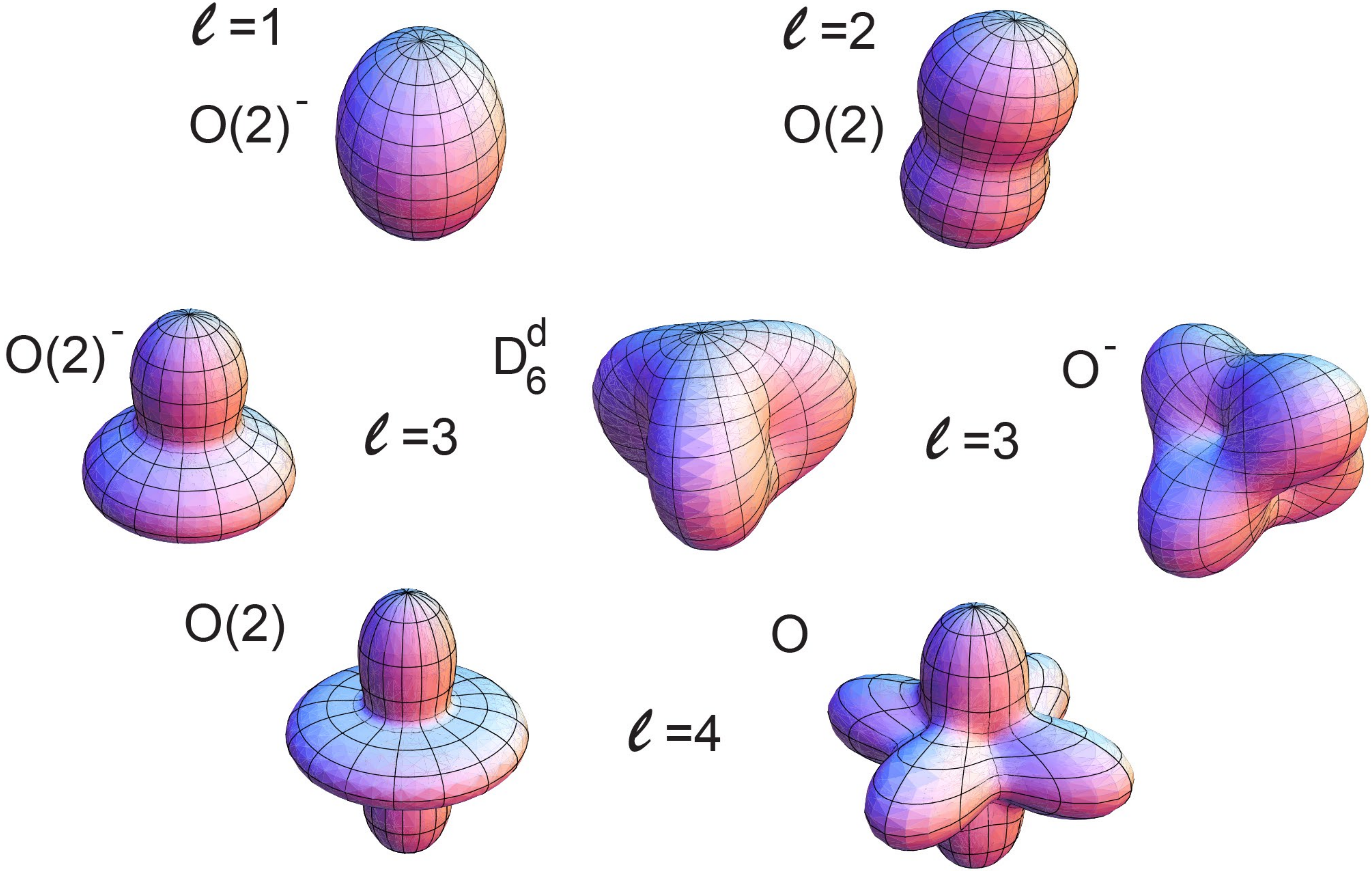
$$\begin{aligned} X^K &:= \{x \mid Kx = x\} \qquad (K\text{-fixed vectors}) \\ G_x &:= \{g \mid gx = x\} \qquad (\text{isotropy of } x) \end{aligned}$$

$$x \in X^K \Rightarrow G_x \geq K$$

Anisotropy inside black hole

joint work with:
Juliette Hell
Brian Smith

Example 7: O(3)-equivariance, stationary



Example 8: equivariant Hopf bifurcation

Setting: $x(t+T) = x(t)$, minimal period T

$$\mathcal{O}_x := \{x(t) | t \in \mathbb{R}\} \qquad \text{(periodic orbit)}$$

$$K := G_{x(t)} \qquad (t\text{-independent!})$$

$$H := \{h \in G \mid h\,\mathcal{O}_x = \mathcal{O}_x\} \qquad \text{(orbit-invariance)}$$

$$\Theta : H \to S^1 := \mathbb{R}/T\mathbb{Z}$$

$$h(x(t)) = x(t + \Theta(h)) \qquad (K = \ker \Theta)$$

$$H^\Theta := \{(h, \Theta(h)) \mid h \in H)\} \qquad \text{(symmetry of } \mathcal{O}_x)$$

$$H^\Theta \leq G \times S^1$$

Lecture Notes in Mathematics

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1309

Bernold Fiedler

Global Bifurcation of Periodic Solutions with Symmetry

Springer-Verlag

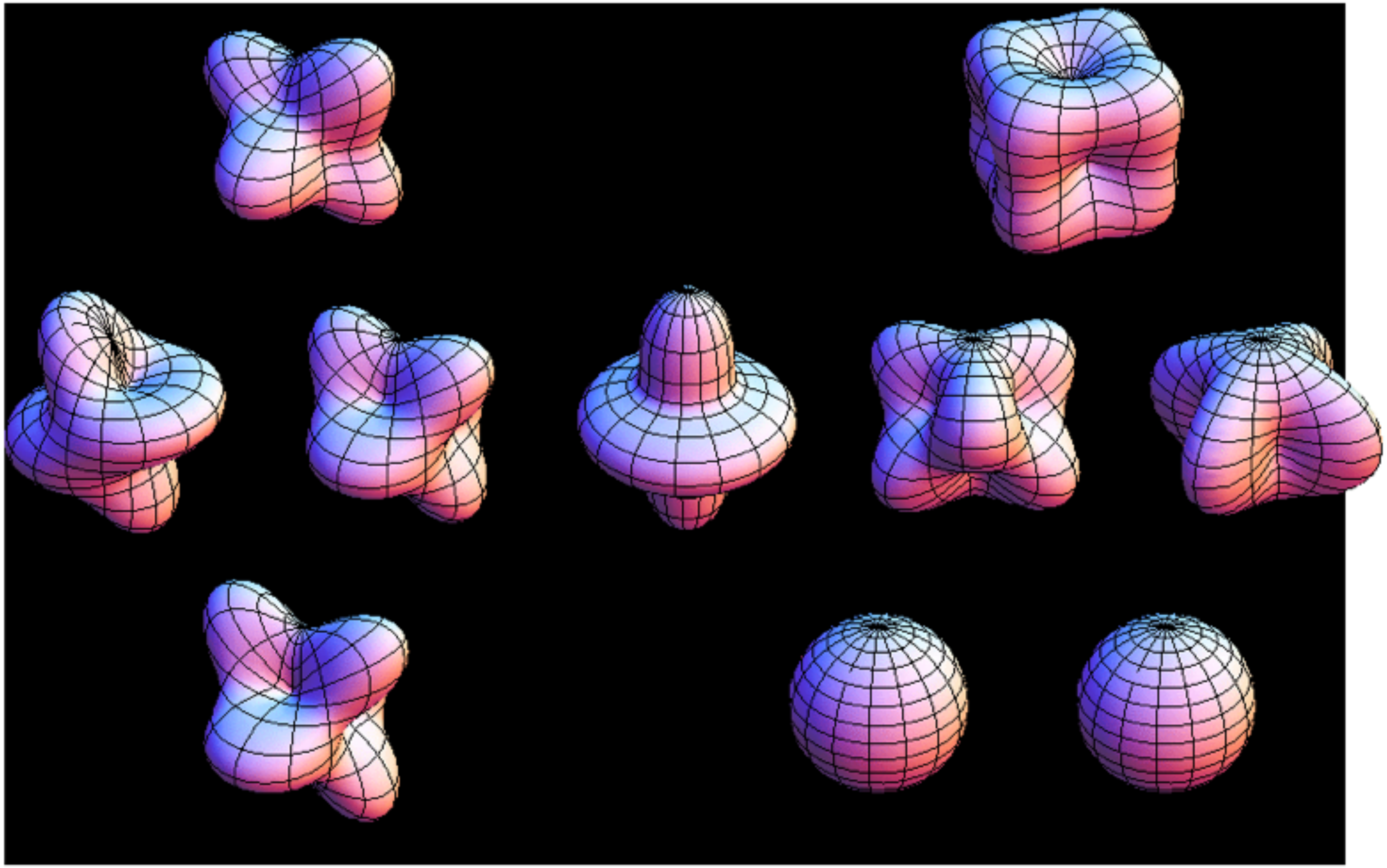
Example 8: equivariant Hopf bifurcation

A4: Schneider, Fiedler

Goal:

$$\mathbf{u}_t = \mathbf{D}\Delta_{S^2}\mathbf{u} + \mathbf{f}(\mathbf{u}) + \begin{cases} b\left(\int_{\xi\in\Xi} \mathbf{u}(\xi x, t-\tau(\xi))d\xi - \mathbf{u}(x,t)\right) & \text{(Fiedler)} \\ b\quad(\Psi\,\mathbf{u}(x-\xi, t-\tau) - \mathbf{u}(x,t)) & \text{(Schneider)} \end{cases}$$

design and delay-stabilize your favorite pattern

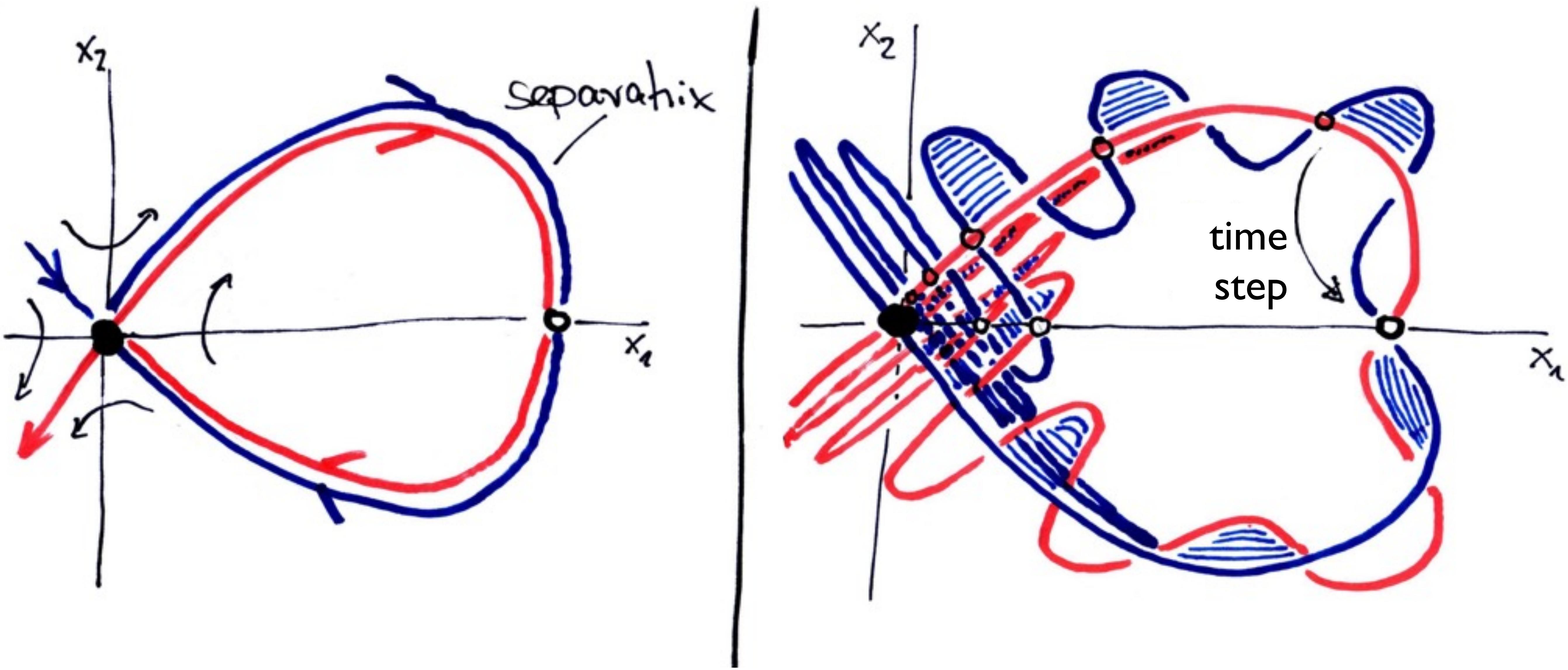


Homoclinic bifurcations

Invisible chaos

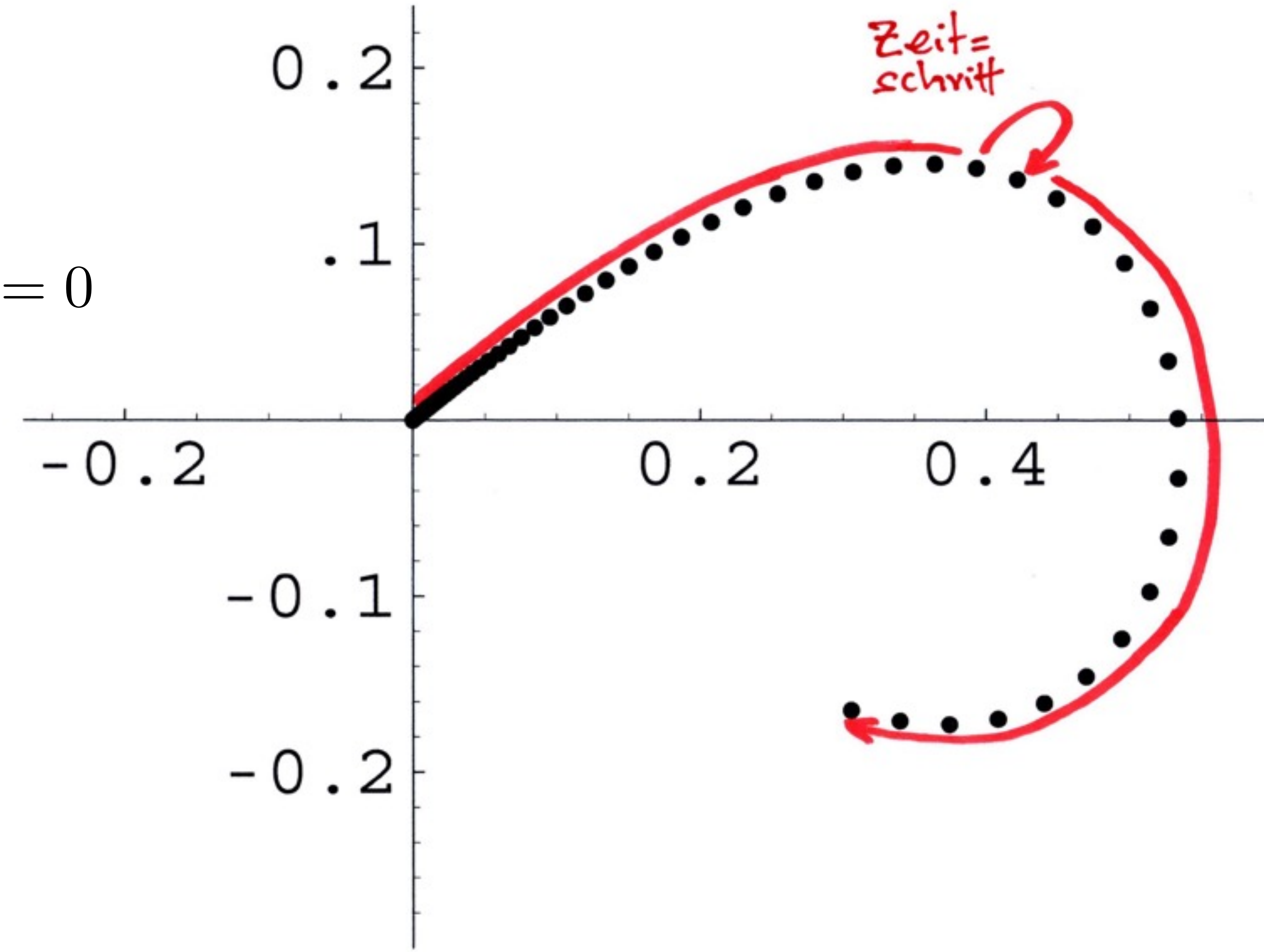
joint work with:
Jürgen Scheurle

homoclinicity: continuous versus discrete

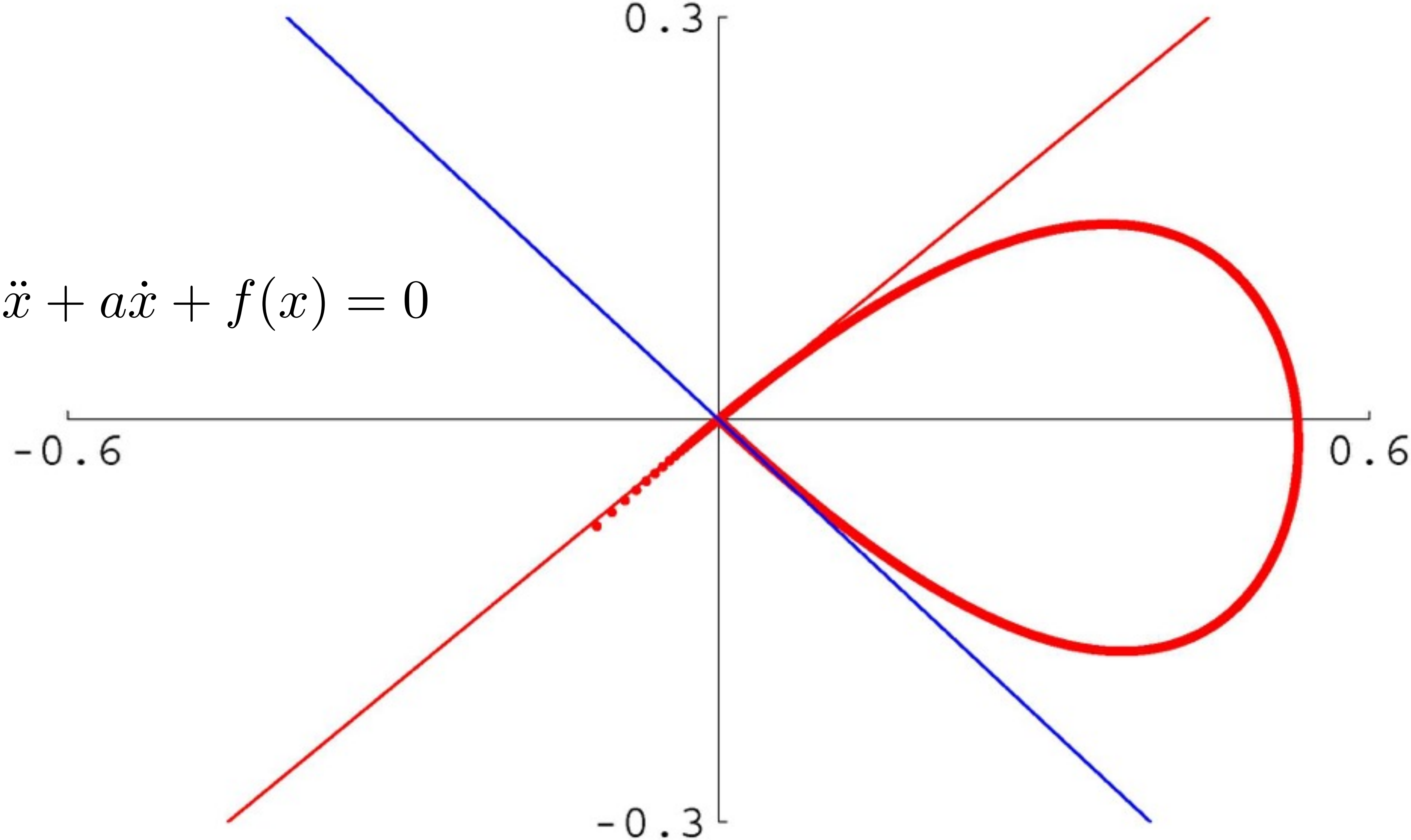


Euler discretization

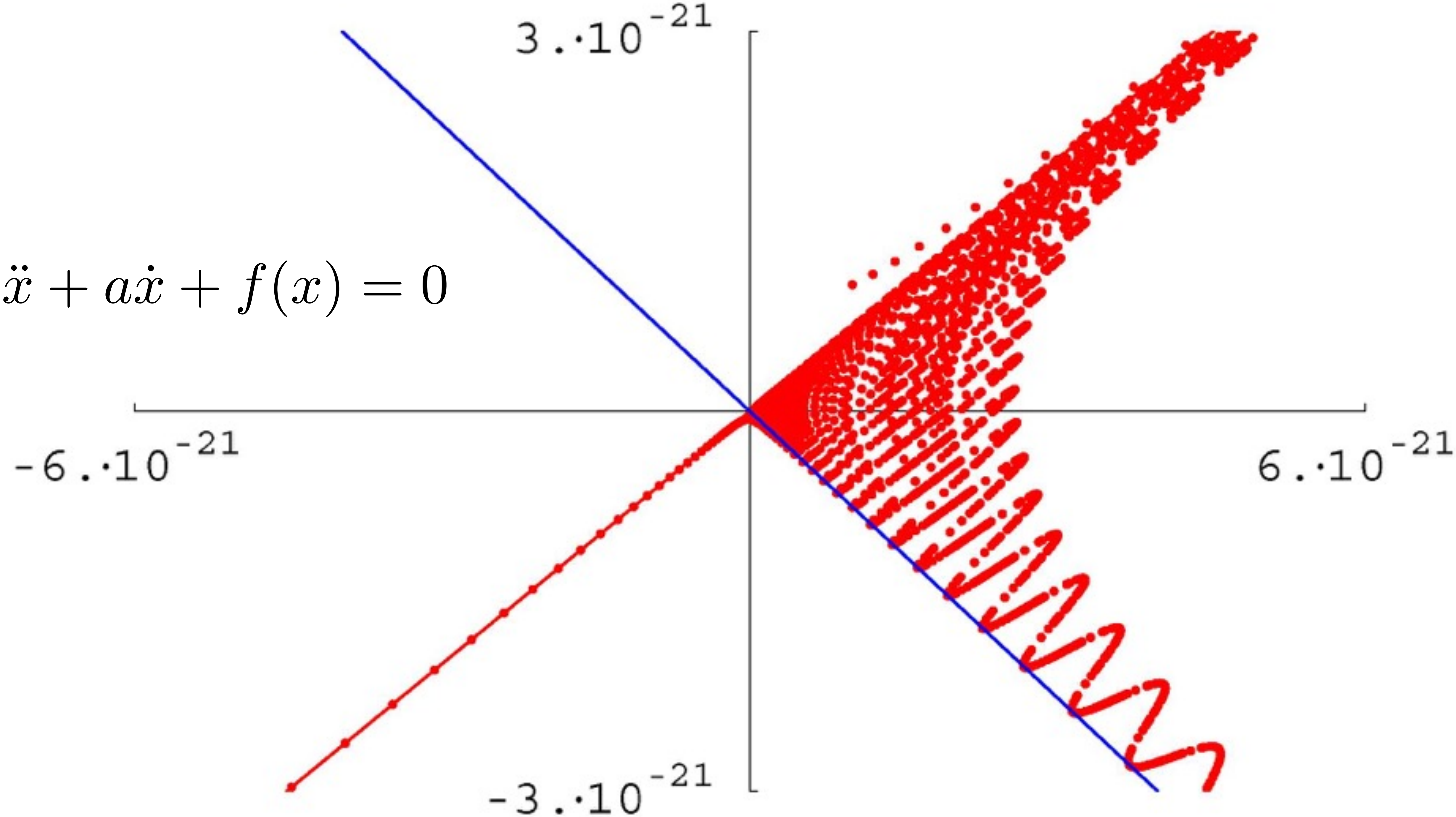
$$\ddot{x} + a\dot{x} + f(x) = 0$$



$$\ddot{x} + a\dot{x} + f(x) = 0$$



a = 0.09016464177938825808032428117539477396..



$$\ddot{x} + a\dot{x} + f(x) = 0$$

Homoclinic doubling

joint work with:
Shui-Nee Chow
Bo Deng

Example 9: homoclinic doubling

Setting: $\dot{x} = A(\lambda_1, \lambda_2)x + \dots,$
homoclinic orbit at $\lambda_2 = 0$,
resonance $\{1, -1\} \subseteq \mathrm{spec}\, A$, at $\lambda_1 = 0$.
[S.-N. Chow, B. Deng, B. Fiedler](1988)

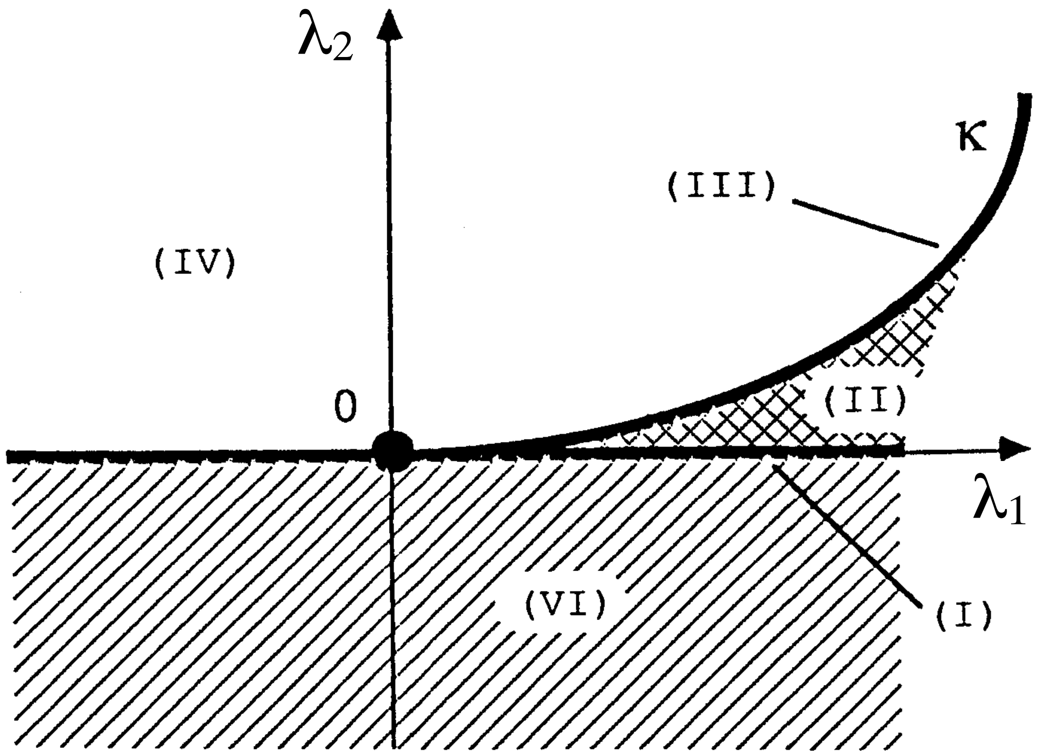
Global homoclinic center manifold,
Shilnikov variables, Poincaré map

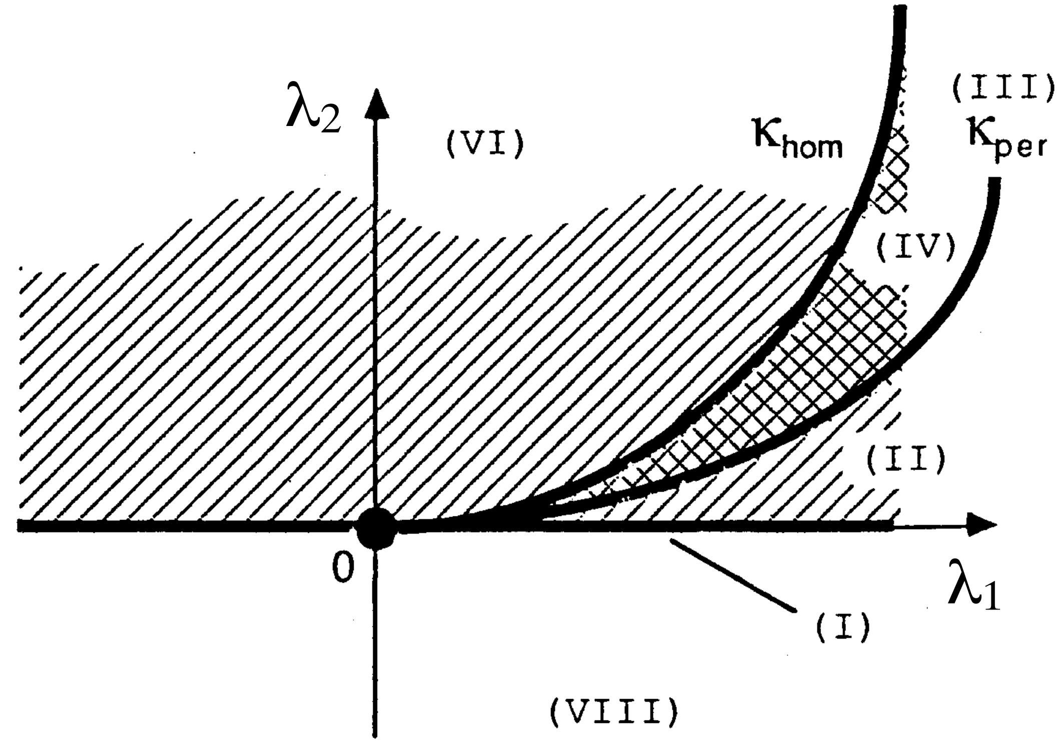
$$r_{n+1} = \lambda_2 + c r_n^{1+\lambda_1}$$

$c > 0$: resonant side switching

$c < 0$: resonant homoclinic doubling

Example 9: homoclinic doubling





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$c > 0$: resonant side switching

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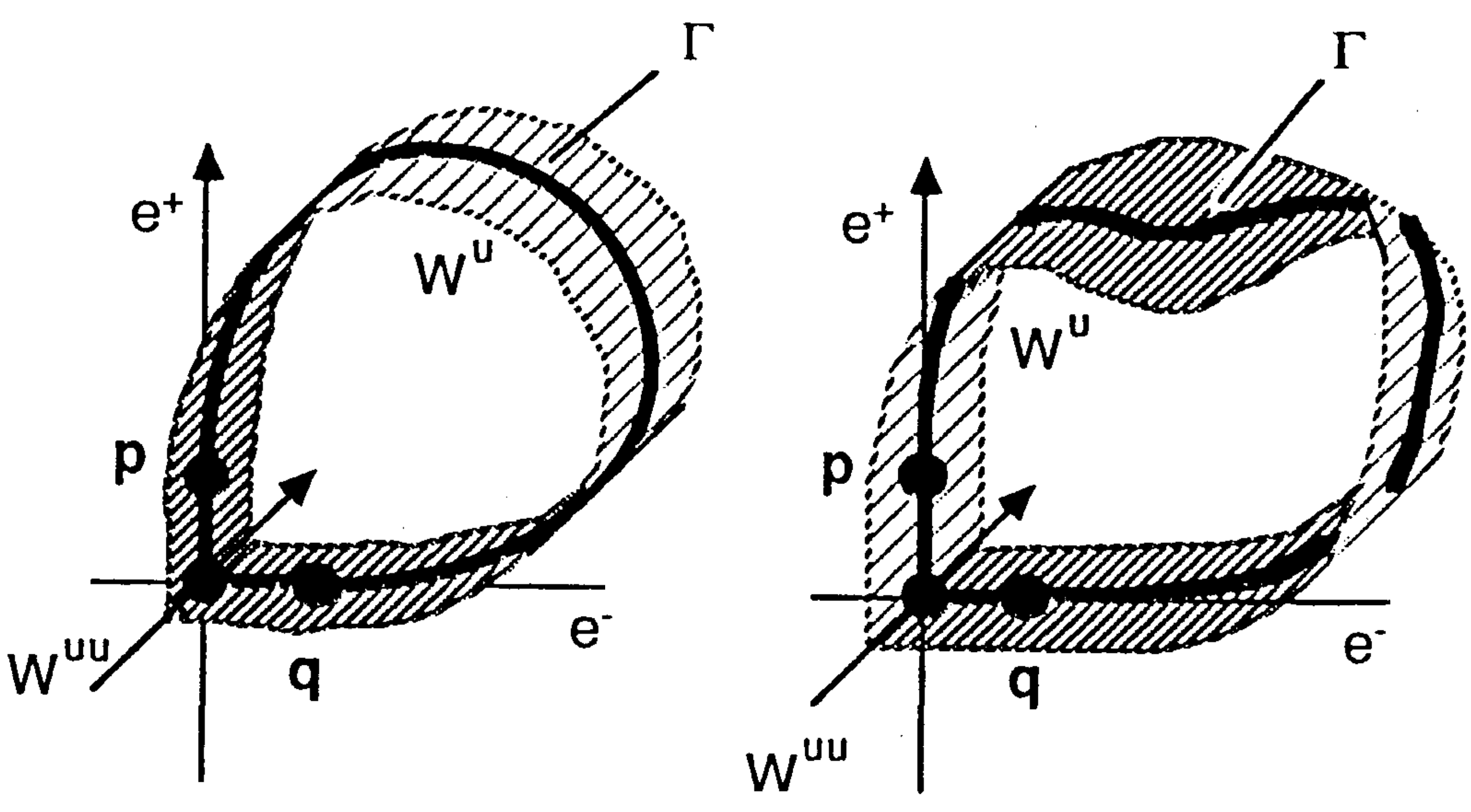
[S.-N. Chow, B. Deng, B. Fiedler](1988)

Global homoclinic center manifold,



much, much more by

Leonid P. Shilnikov
Valentin S. Afraimovich
Vladislav V. Bykov
Sergey S. Gonchenko
Lev L. Lerman
Dmitry V. Turaev
Vassili G. Gelfreich, ...

Bifurcation without parameters

Lecture Notes in Mathematics 2117

Stefan Liebscher

Bifurcation without Parameters

Springer

Example 10: without parameters

Setting: lines, planes, . . . , of trivial equilibria
[S. Liebscher](2015)

- **with parameter λ**

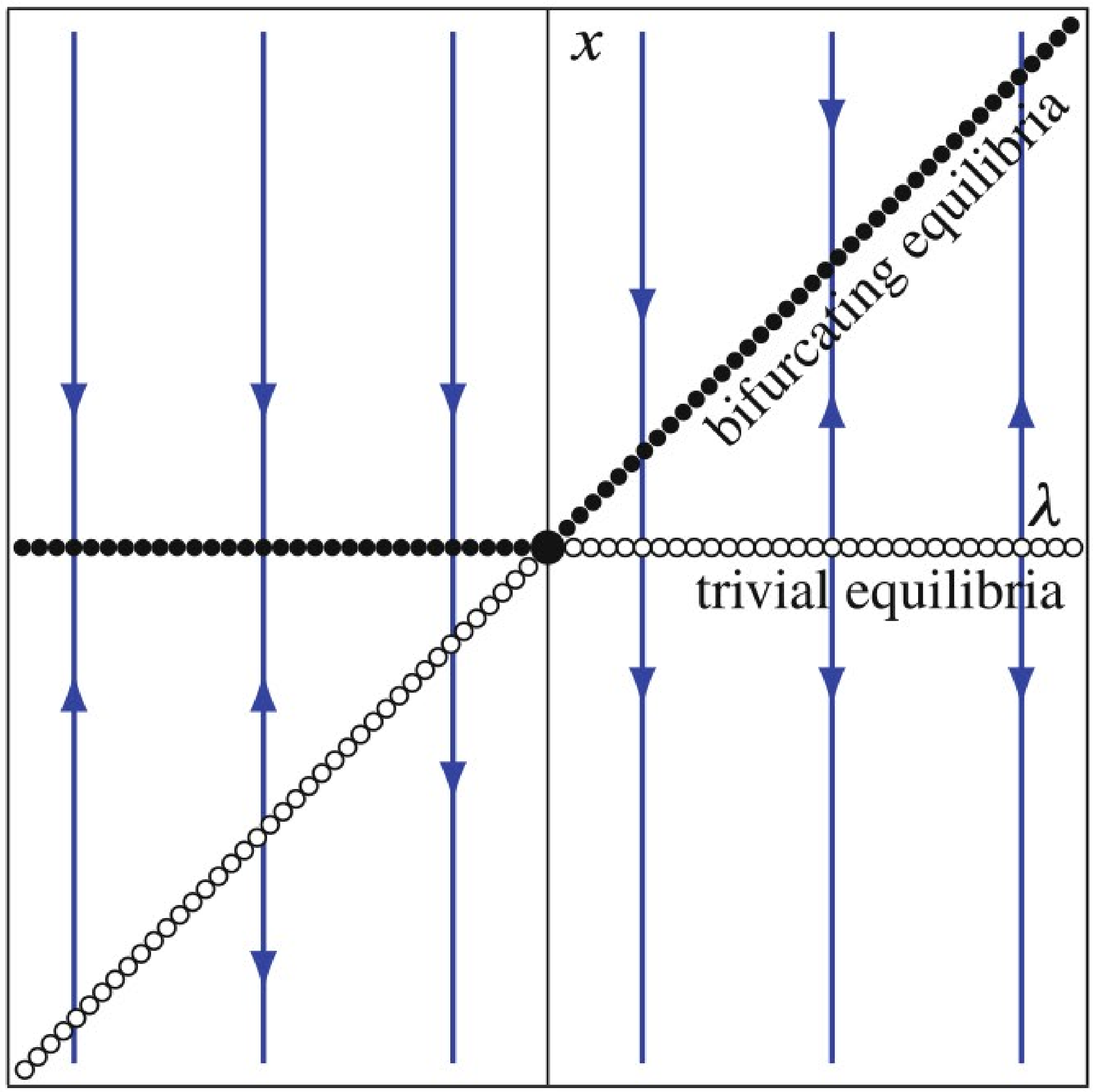
$$\dot{x} = f(x, \lambda) = A(\lambda)x + \ldots\,, \qquad f(0, \lambda) \equiv 0$$
$$\dot{\lambda} = g(x, \lambda)\,, \qquad \textbf{all } g(x, \lambda) \equiv 0$$

- **without parameter**

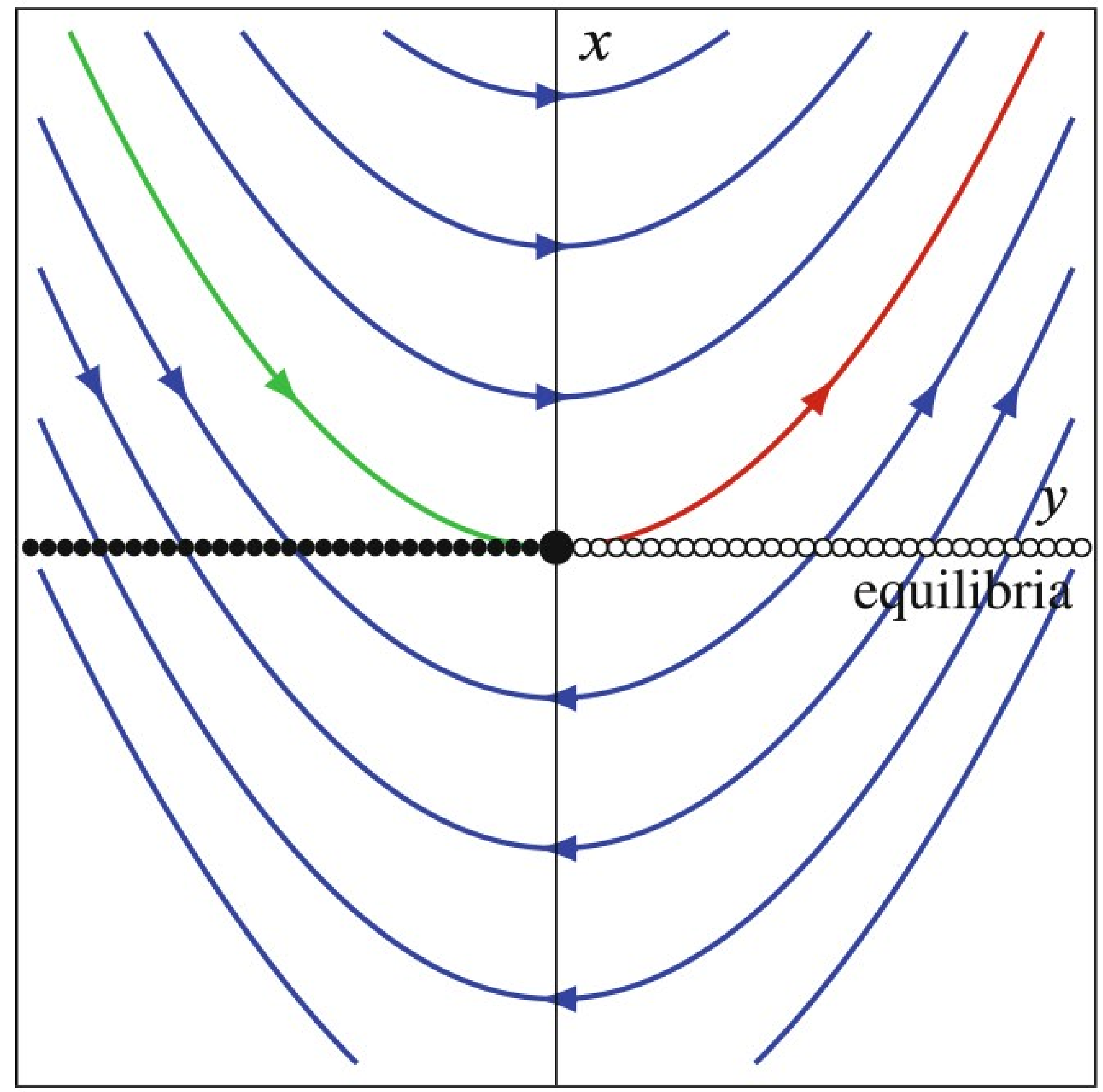
$$\dot{x} = f(x, y) = A(y)x + \ldots\,, \qquad f(0, y) \equiv 0$$
$$\dot{y} = g(x, y)\,, \qquad \textbf{only } g(0, y) \equiv 0$$

Example 10: stationary without parameters

parameter λ

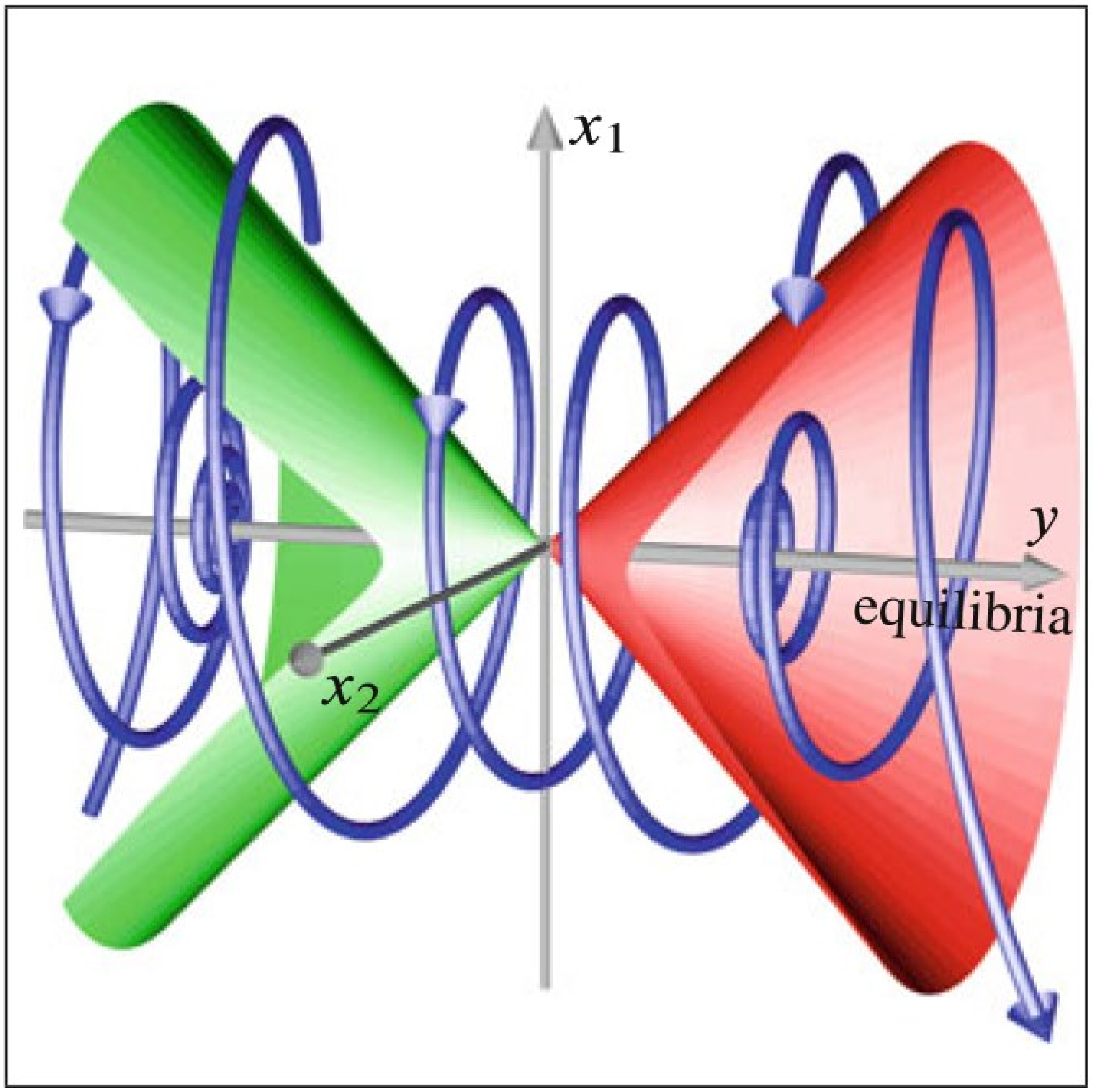


no parameter

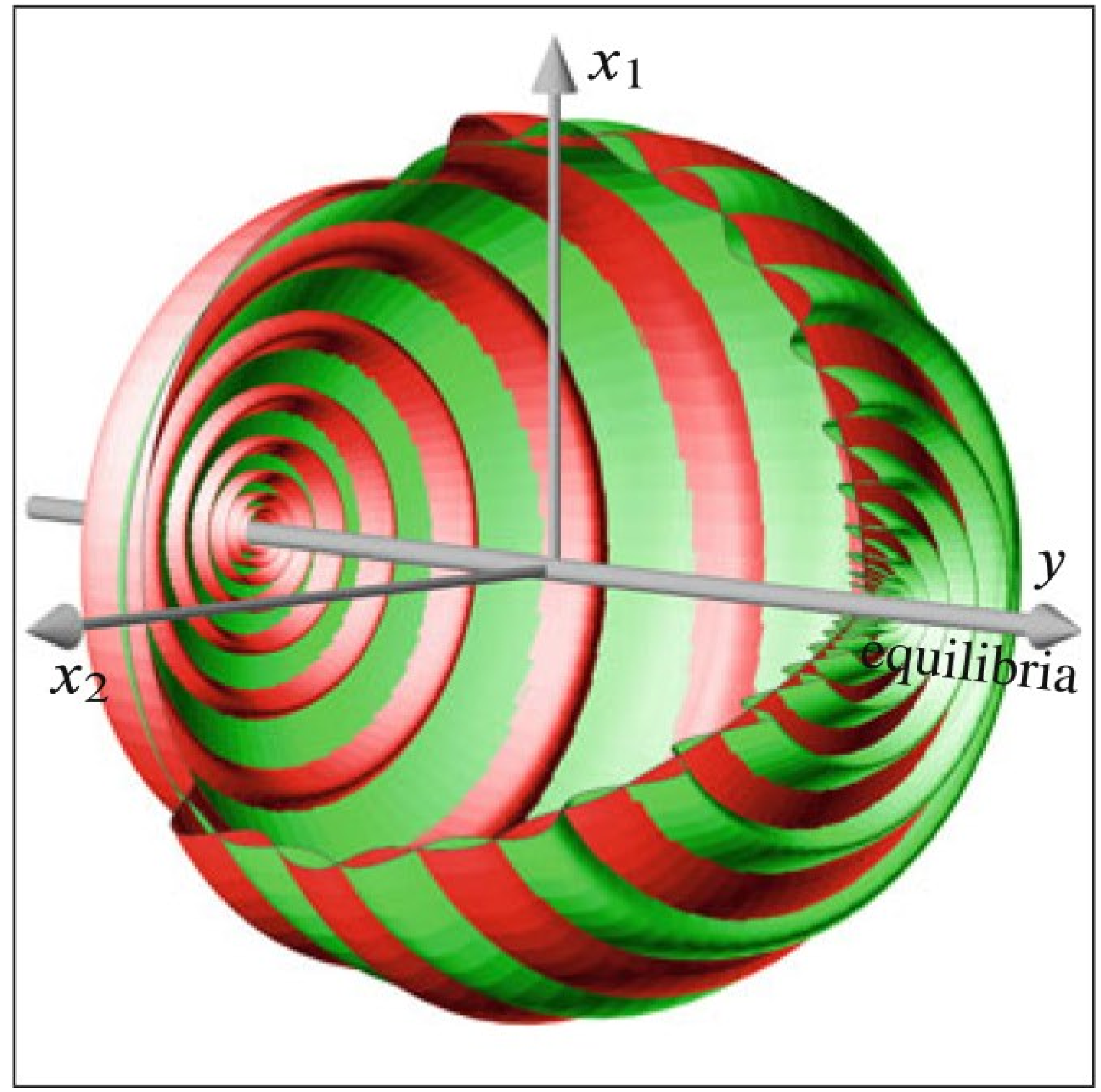


Example 10: Hopf without parameters

Hopf, hyperbolic



Hopf, elliptic

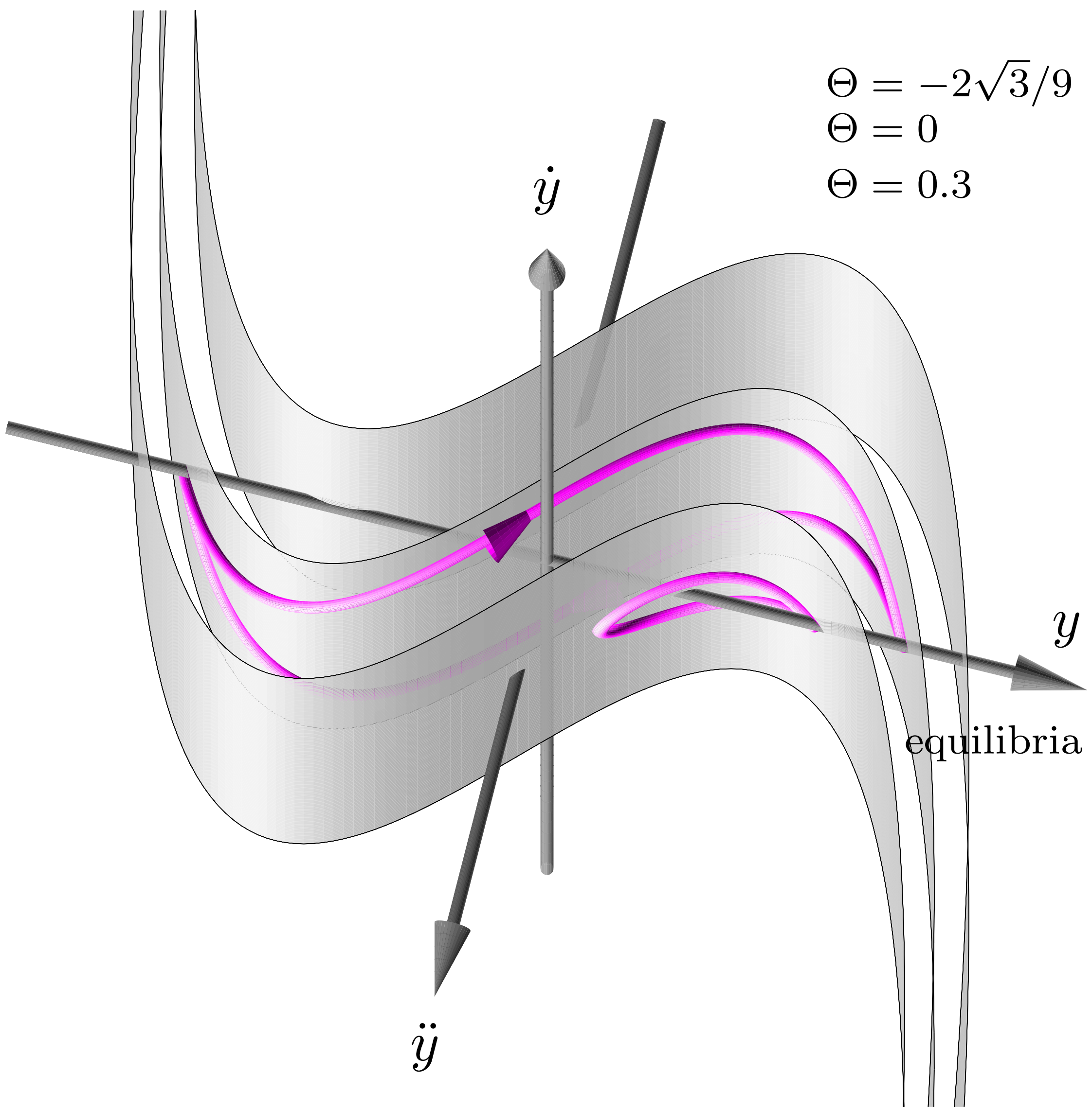


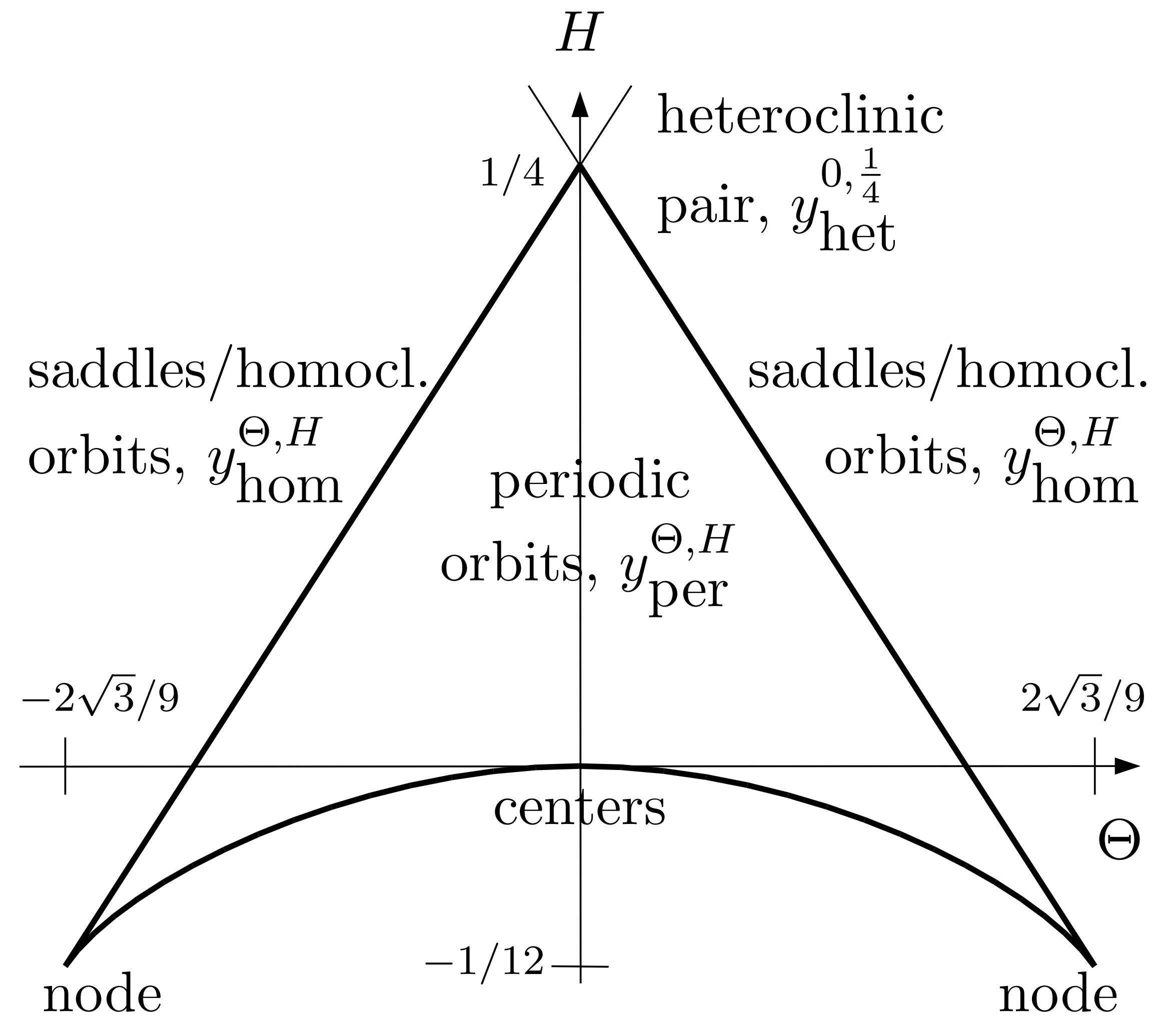
Kolmogorov fluid flows

Andrey N. Kolmogorov (1961)
[Afendikov, Fiedler, Liebscher](2007,2008,2011)

Example 10: Bogdanov-Takens without parameters

Kolmogorov fluid flow in 2-channel





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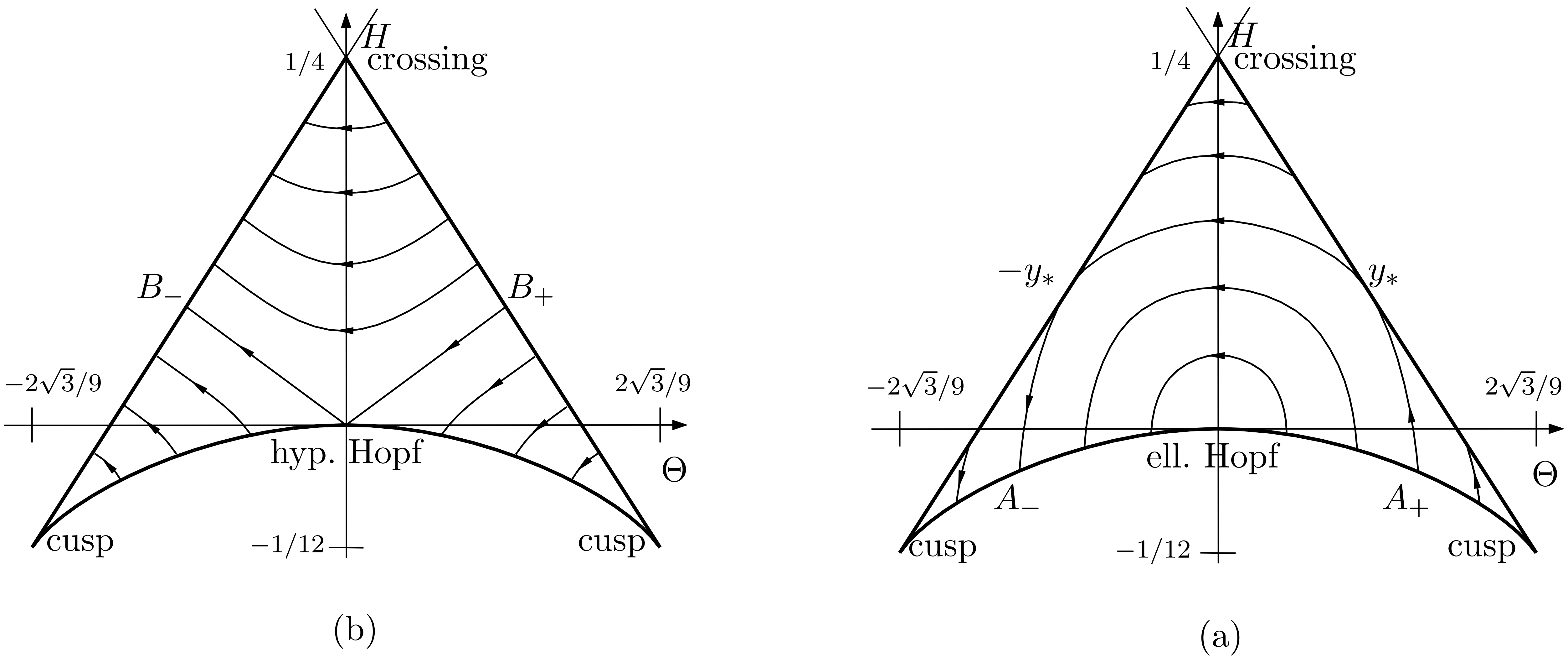

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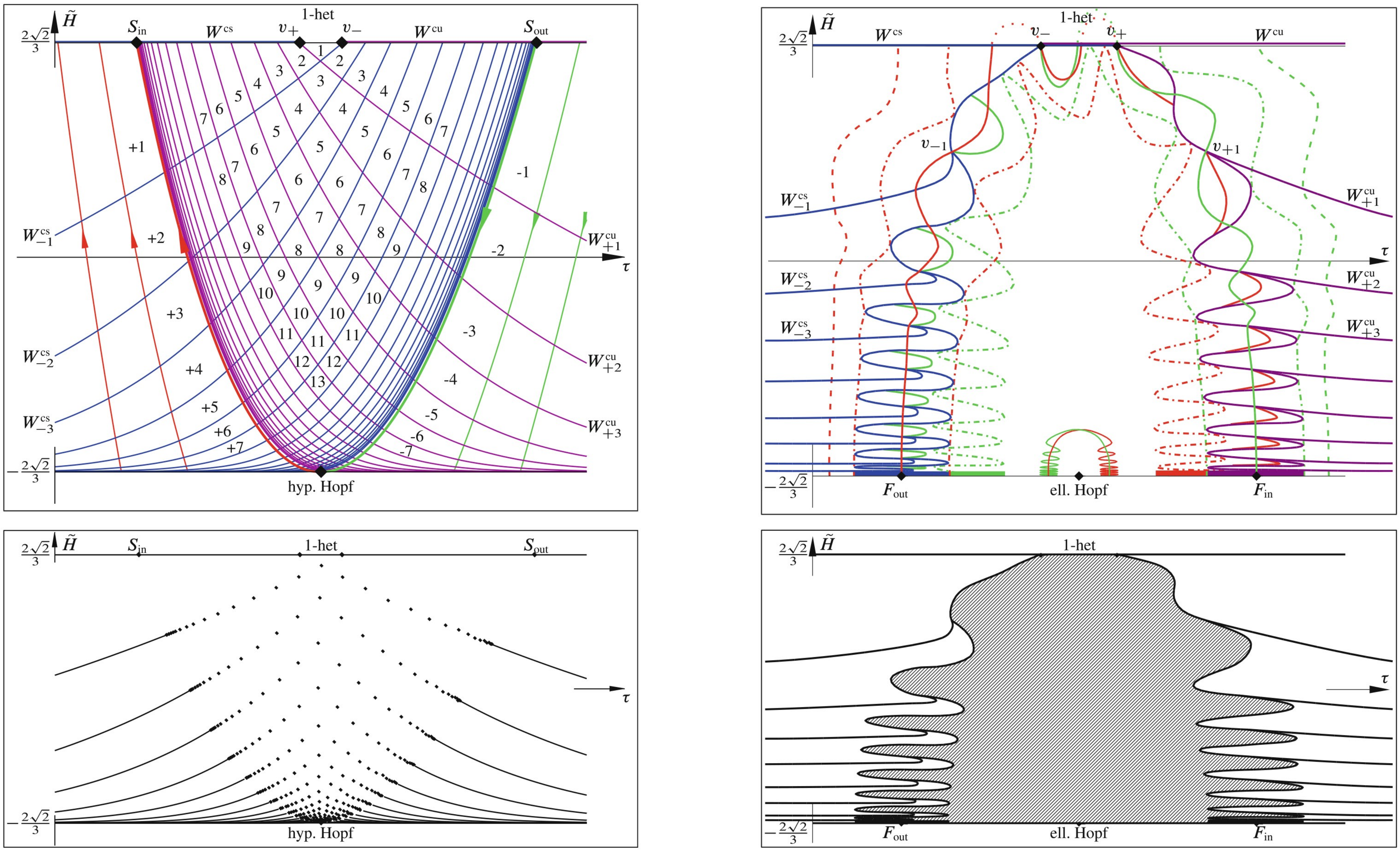

Conclusions

Summary

- Ex1: flow box, Ex2: hyperbolic equilibrium - no bif.s
- Ex3: saddle-node, transcritical, pitchfork bif.s
- Ex4: period doubling bifurcation
- Ex5: Hopf bifurcation
- Ex6: Bogdanov-Takens bifurcation
- Ex7: stationary symmetry breaking
- Ex8: equivariant Hopf
- Ex9: homoclinic doubling and invisible chaos
- Ex10: bifurcation without parameters
- Mathematical justifications