Homework Assignments

Bifurcations: Theory and Applications

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 45: Consider the non-autonomous linear system

$$\dot{z} = A(t)z,$$

with

$$A(t) \coloneqq \begin{cases} A_+, & \text{if } t \geq 0, \\ A_-, & \text{if } t < 0, \end{cases}$$

where $A_\pm$ are hyperbolic. Prove that

$$\begin{array}{rcl} \mathcal{L} : H^1 & \to & L^2 \\ z(t) & \mapsto & (\mathcal{L}z)(t) \coloneqq \dot{z}(t) - A(t)z(t) \end{array}$$

is a Fredholm operator with index given by the difference unstable dimensions of the hyperbolic matrices $A_\pm$.

Problem 46: Consider the non-autonomous linear system

$$\dot{z} = A(t)z.$$

Assume the matrices $A(t)$ depend smoothly on $t \in \mathbb{R}$ (L^∞ is sufficient), with hyperbolic limits $A_\pm$ for $t \to \pm\infty$. Prove that

$$\begin{array}{rcl} \mathcal{L} : H^1 & \to & L^2 \\ z(t) & \mapsto & (\mathcal{L}z)(t) \coloneqq \dot{z}(t) - A(t)z(t) \end{array}$$

is Fredholm, and the index coincides with the index from problem 45.

Problem 47: [This exercise counts as two]
While enduring the hardships of this course on Dynamical Systems, you may have come across the claim that "the pendulum with periodic forcing is chaotic", more than once. We've got good news for you, now you have the tools to give a formal prove!

Consider the pendulum equation

$$\ddot{x} + \sin x = 0,$$

(i) Show that

$$x(t) = 2\arctan(\sinh t),$$

provides, explicitly, a homoclinic orbit from $x = -\pi$ to $x = \pi \equiv -\pi \pmod{2\pi}$.

(ii) Show that $\Psi(t) := \nabla H(x(t), \dot{x}(t))$ solves the adjoint variational equation, where H denotes the Hamiltonian energy

$$H(x, \dot{x}) := \tfrac{1}{2}\dot{x}^2 - \cos x.$$

(iii) Derive the Melnikov function for the general perturbation

$$(1)_\varepsilon \qquad \ddot{x} + (1 + \varepsilon g(t))\sin x = 0,$$

where g is a smooth 2π-periodic function.

(iv) Show that the stroboscope map of $(1)_\varepsilon$ possesses a transverse homoclinic orbit (and hence chaotic dynamics), for $g(t) = \sin(t + \beta)$ with a suitable β, and small $\varepsilon > 0$.

Problem 48: [Extra credit] Let X and Y be Banach spaces. Prove that the set of compact linear operators from X to Y is closed in the space of bounded linear operators $L(X, Y)$, equipped with the operator norm.