Homework Assignments

Bifurcations: Theory and Applications

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due date: Wednesday, November 6, 2019, 12:00

Problem 5: [Center manifold theorem with parameters]
Use the center manifold theorem to prove the parametric version:
For $x \in X = \mathbb{R}^N$, parameters $\lambda \in \Lambda = \mathbb{R}^k$, and $f \in C^1(\Lambda \times X, X)$, consider the ODE

$$\dot{x} = f(\lambda, x) = Ax + g(\lambda, x), \text{ with } g \text{ bounded in } \lambda \text{ and } x.$$

Then there exists $\delta > 0$ such that for $\sup_{x \in \mathbb{R}^N} |D_{(\lambda,x)} g(\lambda, x)| < \delta$ the following holds:

(i) The sets $M^c(\lambda) := \{x_0 \in \mathbb{R}^N \mid \sup_{t \in \mathbb{R}} |x^h(\lambda, t)| < \infty\}$ are invariant C^1-manifolds, for each λ, which are given as the graphs of a family of bounded C^1 functions

$$\eta : \Lambda \times E^c \to E^h.$$

Here $x(\lambda, t) := \varphi_t(\lambda, x_0)$ is the solution of the ODE at parameter value λ with initial condition x_0. The spaces E^c, E^h are the center and hyperbolic parts of the spectral splitting induced by A, respectively.

(ii) $M^c(\lambda)$ is the only such invariant BC^1 manifold.

Problem 6: For $x \in X = \mathbb{R}^N$ consider the ODE

$$\dot{x} = x^2,$$
$$\dot{y} = -y.$$

(i) Suppose $M^c := \{(x, \eta(x)) \mid x \in \mathbb{R}\}$ with $\eta \in C^1(\mathbb{R}, \mathbb{R})$ is a flow-invariant manifold. Show $\eta(x) = 0$ for all $x \geq 0$.

(ii) Show that the graphs of

$$\eta_\alpha(x) := \begin{cases} \alpha e^{1/x} & \text{if } x < 0, \\ 0 & \text{elsewhere.} \end{cases}$$

define smooth invariant manifolds parametrised over the center direction E^c. How is this compatible with the uniqueness part of the center manifold theorem seen in the lecture?

Problem 7: Consider the smooth iteration on $\mathbb{R}^2$

$$x_{n+1} = F(x_n) = Bx_n + \mathcal{O}(|x_n|^2) := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x + \mathcal{O}(|x_n|^2).$$

As usual, let $W_m := \mathrm{Ker}(B^T[\cdot]_m)$, where $B^T[\cdot]_m$ is the restriction to the space of homogeneous polynomials of degree m of the commutator

$$B^T[\phi](x) := B^T\phi(x) - \phi(B^T x),\ \phi \in C^\infty(\mathbb{R}^2, \mathbb{R}^2).$$

(i) Derive necessary and sufficient conditions on $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ under which there exists $m \geq 2$ such that $W_m \neq 0$.

(ii) Interpret your result: under which conditions on (λ_1, λ_2) and in which sense, can the iteration F be "linearized" (dissertation of Henri Poincaré)?

Problem 8: Consider the smooth iteration on $\mathbb{R}$

$$x_{n+1} = F(x_n) = -x_n + \mathcal{O}(|x_n^2|), \qquad x \in \mathbb{R}.$$

(i) Show that the B^T normal form

$$y_{n+1} = -y_n + b_3 y_n^3 + b_5 y_n^5 + \dots$$

is odd, to any finite order.

(ii) [Extra credit] Determine the stability of the fixed points $y = 0$ and $x = 0$, respectively, depending on $a_2 := F''(0)/2$ and $a_3 := F'''(0)/6$.