Homework Assignments

Bifurcations: Theory and Applications

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Problem 13: Consider the smooth vector field

$$\dot{x} = f(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x + \mathcal{O}(|x|^2)$$

Compute $W_m = \mathrm{Ker}(\mathrm{ad}_m A^T)$, up to order 4.

Problem 14: Consider the smooth ODE in $\mathbb{R}^N$

$$\dot{x} = f(x) = Ax + \mathcal{O}(|x|^2),$$

with time-1 map denoted Φ_1. Derive non-resonance conditions for the iteration

$$x_{n+1} = \Phi_1(x_n) = Bx_n + \mathcal{O}(|x_n|^2),$$

where $B := \exp(A) = \mathrm{diag}\{\lambda_1, \ldots, \lambda_N\}$.

Problem 15: Consider the Lie algebra $\mathfrak{so}(3)$ of skew-symmetric real (3×3) matrices. Show that the map

$$\mathfrak{so}(3) \longrightarrow \mathbb{R}^3, \qquad \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \longmapsto \begin{pmatrix} -c \\ b \\ -a \end{pmatrix}$$

transforms the Lie bracket

$$[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a}$$

into the vector product on $\mathbb{R}^3$. Give a geometric interpretation.

Analogously, show that the map

$$\mathfrak{su}(2) \longrightarrow \mathbb{R}^3, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} ai & ib + c \\ ib - c & -ai \end{pmatrix} \longmapsto \begin{pmatrix} -c \\ b \\ -a \end{pmatrix},$$

defined on the Lie algebra $\mathfrak{su}(2)$, transforms the Lie bracket into the vector product on $\mathbb{R}^3$.

Find an isomorphism between the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.

Problem 16: Consider the truncated normal form for the Arnol'd-Takens-Bogdanov bifurcation

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = ax_1^2 + bx_1x_2,$$

where $a, b \neq 0$. Discuss how one can assume $a = b = 1$, without loss of generality, by making use of spatial and temporal rescalings (which can involve changes in the time direction).