Homework Assignments

Bifurcations: Theory and Applications

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Problem 33: [Euler's beam, revisited]
Consider the map

$$f(\lambda, u) := u'' + \lambda \sin u$$

on the space of 2π-periodic functions, $f : \mathbb{R} \times C^2(S^1, \mathbb{R}) \to C^0(S^1, \mathbb{R})$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

(i) Determine the subgroup G of $\mathcal{L}(C^0(S^1, \mathbb{R}), C^0(S^1, \mathbb{R}))$ under which $f(\lambda, \cdot)$ is equivariant.

(ii) Consider $\kappa \in G$ such that

$$(\kappa u)(x) = u(-x), \qquad K := \langle \kappa \rangle.$$

Show that the fixed-point subspace $C^2(S^1, \mathbb{R})_K$ can be identified with the subspace of functions with Neumann ($\{u'(0) = u'(\pi) = 0\}$) boundary conditions.

Problem 34: Based on the results from Problem 33, part (ii), find branches of non-trivial solutions $f(\lambda, u) = 0$ bifurcating from the trivial solution $u \equiv 0$. Which isotropy do the bifurcating solutions possess?

Problem 35: [Lyapunov center theorem]
Consider the Hamiltonian system

$$\dot{x} = J\nabla H(x), \qquad J = \begin{pmatrix} 0 & \mathrm{Id}_N \\ -\mathrm{Id}_N & 0 \end{pmatrix}, \qquad H \in C^2(\mathbb{R}^{2N}, \mathbb{R}).$$

Let the origin, $x = 0$, be an equilibrium, $\nabla H(0) = 0$. Suppose that the linearization $JD^2H(0)$ has a pair of algebraically simple eigenvalues $\pm\mathrm{i}$, and that all other eigenvalues have non-vanishing real parts.

Prove that in a neighborhood of the origin there is an invariant 2-dimensional manifold filled with periodic orbits.

Hint: Consider the system $\dot{x} = (\lambda \mathrm{Id}_{2N} + J)\nabla H(x)$ with a real parameter λ and discuss the Hopf bifurcation taking place at $\lambda = 0$.

Problem 36: Let ρ be a representation of a group Γ on a Banach space X. Let $K < \Gamma$ be a proper subgroup of Γ. We call K an *isotropy subgroup* of Γ, if $K < \Gamma$ and there exists $x \in X$ such that $K = \Gamma_x$ is the isotropy of x. An isotropy subgroup is called *maximal*, if it is not a proper subgroup of an isotropy subgroup.
For any isotropy subgroup $K < \Gamma$ show

$$\text{If } \dim X_K = 1 \text{ then } K \text{ is a maximal isotropy subgroup.}$$