Extra Homework Assignments

Bifurcations: Theory and Applications

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Only attempted exercises will be discussed in tutorials. All exercises are for extra credit: the points you obtain will count towards your "Übungsschein", but we will not use these questions to calculate the 25% threshold of the "Übungsschein".

Weihnachtsaufgabe 1: Let ρ be a real, finite-dimensional, irreducible representation of a group Γ on a real vector space X. Prove or disprove:

If $\dim X$ is odd, then ρ is absolutely irreducible.

Weihnachtsaufgabe 2: Consider the heat equation

$$\partial_t u(t,x) = \partial_{xx} u(t,x),\ x \in \mathbb{R},\ t > 0.$$

Find, explicitly, a solution by using the Ansatz

$$u(t,x) = G(t,x) := \tfrac{1}{\sqrt{t}} v\left(\tfrac{x}{\sqrt{t}}\right),\ x \in \mathbb{R},\ t > 0,$$

where $v \in C^2(\mathbb{R},\mathbb{R}) \cap L^2(\mathbb{R})$, $v(0) = 1/\sqrt{4\pi}$.

[Extra credit] Show that the solutions of the nonhomogeneous heat equation

$$\begin{aligned} \partial_t u(t,x) &= \partial_{xx} u(t,x),\ x \in \mathbb{R},\ t > 0, \\ u(0,x) &= f,\ f \in L^2(\mathbb{R}), \end{aligned}$$

are given by $u(t,x) = (G(t,\cdot) * f)(x)$, $t > 0$, where $*$ denotes convolution. Which initial value problem does G solve?
Hint: Consider the Fourier transform of the heat equation.

Weihnachtsaufgabe 3: Show that $\rho(\gamma) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$(\rho(\gamma)u)(x) = \tfrac{1}{\sqrt{4\pi\gamma}} \int_{\mathbb{R}} \exp\left(\tfrac{-(x-y)^2}{4\gamma}\right) u(y)\mathrm{d}y,\ \gamma > 0,\ x \in \mathbb{R},$$

is a representation of the semigroup $([0,+\infty),+)$ on $L^2(\mathbb{R})$. Is it norm continuous? Is it strongly continuous?

Weihnachtsaufgabe 4: Let $G \leq SO(3)$ denote the subgroup of orientation preserving real orthogonal 3×3-matrices which map the cube $[-1, 1]^3$ onto itself, as a set (but not necessarily fixing each point of the cube).

(i) Characterize G as a group Γ of permutations, e.g. of the vertices of the cube. In other words $G = \rho(\Gamma) \leq SO(3)$, for a certain representation ρ of the finite group Γ of permutations, by rotations in $\mathbb{R}^3$. Try to describe the group Γ as simply as possible.

(ii) Is the representation irreducible? Is it absolutely irreducible?

(iii) What happens if we replace the cube $\max_n |x_n| \leq 1$ by the octahedron $\sum_n |x_n| \leq 1$?

Weihnachtsaufgabe 5: In the setting of Weihnachtsaufgabe 4 determine all isotropy subgroups of G, up to conjugacy, and their fixed-point subspaces.

[Extra credit] How do the results of Weihnachtsaufgaben 4 and 5 change for $G \leq O(3)$?

Weihnachtsaufgabe 6: Derive the formula for the Laplacian

$$\Delta := \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 = \frac{1}{r^2} \left(\partial_r (r^2 \partial_r) + \tfrac{1}{\sin \vartheta} \partial_\vartheta ((\sin \vartheta)\, \partial_\vartheta) + \tfrac{1}{\sin^2 \vartheta} \partial_\varphi^2 \right),$$

in standard polar coordinates on $\mathbb{R}^3$

$$\begin{aligned} x_1 &= r \sin \vartheta \cos \varphi, \\ x_2 &= r \sin \vartheta \sin \varphi, \\ x_3 &= r \cos \vartheta. \end{aligned}$$

Use the variational formulation

$$\int_{\mathbb{R}^3} \Delta u, v = - \int_{\mathbb{R}^3} \nabla u, \nabla v,$$

for smooth u, v with compact support.

Weihnachtsaufgabe 7: The gradient ∇_x of $u \in C^1(\mathbb{R}^N, \mathbb{R})$ is the only vector satisfying

$$D_x u(x) y = \langle \nabla_x u, y \rangle\,, \text{ for all vectors } y \in \mathbb{R}^N.$$

Consider the usual representation ρ of the orthogonal group $\Gamma = O(N)$ on $C^1(\mathbb{R}^N, \mathbb{R})$, via $(\rho(\gamma)u)(x) = u(\gamma^{-1} x)$ and derive the coordinate transformation of the gradient

$$\nabla_{\gamma^{-1} x} = \gamma^T \nabla_x.$$

Weihnachtsaufgabe 8: Consider $u \in L^2(S^2, \mathbb{R})$. The Laplacian, Δu, and gradient, ∇u, are defined (variationally) as the operators that satisfy

$$\int_{S^2} \Delta u v = -\int_{S^2} \nabla u \cdot \nabla v = \int_{S^2} u \Delta v \text{ for all smooth functions } v.$$

Given the usual representation ρ of $\Gamma = O(3)$ on $L^2(S^2, \mathbb{R})$, $(\rho(\gamma)u)(x) = u(\gamma^{-1}x)$. Show the equivariance of the Laplacian under orthogonal transformations, i.e.

$$\rho(\gamma)(\Delta u) = \Delta(\rho(\gamma)u).$$

Note: Δv, ∇v are the usual Laplacian and gradient for smooth functions.

[Extra credit] What about the Laplacian on $L^2(B^3, \mathbb{R})$? Here B^3 denotes the unit ball in $\mathbb{R}^3$ and

$$\int_{B^3} \Delta u v = -\int_{B^3} \nabla u \cdot \nabla v = \int_{B^3} u \Delta v \text{ for all smooth functions } v.$$

Weihnachtsaufgabe 9: Determine the spectrum of the Laplace-Beltrami operator Δ_{S^2} on the 2-sphere from the kernel of the Laplacian Δ on the space H_ℓ of homogeneous complex valued polynomials $u : \mathbb{R}^3 \to \mathbb{C}$. Use the Ansatz $u = r^\ell v(\vartheta, \varphi)$ in polar coordinates. Show that the eigenspaces V_ℓ of Δ_{S^2}, given by the restriction of the polynomials in $\operatorname{Ker} \Delta \leq H_\ell$ to S^2, are spanned by the spherical harmonic functions

$$Y_{\ell m} = P_m^\ell(\cos\vartheta) \exp(\mathrm{i} m \varphi).$$

Here P_m^ℓ denote the associated Legendre polynomials. (Yes, look up those definitions!)

Weihnachtsaufgabe 10:

The Poisson bracket of $G, H \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ is defined as

$$\{H, G\} \coloneqq \langle \nabla G, J \nabla H \rangle, \qquad J = \begin{pmatrix} 0 & \mathrm{Id}_N \\ -\mathrm{Id}_N & 0 \end{pmatrix}.$$

The Lie bracket of two vector fields $g, h \in C^\infty(\mathbb{R}^{2N}, \mathbb{R}^{2N})$ is defined as

$$[h, g](x) \coloneqq Dh(x)g(x) - Dg(x)h(x).$$

Show that:

(i) $(C^\infty(\mathbb{R}^{2N}, \mathbb{R}), \{\cdot,\cdot\})$ and $(C^\infty(\mathbb{R}^{2N}, \mathbb{R}^{2N}), [\cdot,\cdot])$ are Lie algebras.

(ii) The map

$$\begin{array}{rcl} X_\cdot : C^\infty(\mathbb{R}^{2N}, \mathbb{R}) & \to & C^\infty(\mathbb{R}^{2N}, \mathbb{R}^{2N}) \\ H & \mapsto & X_H \coloneqq J\nabla H, \end{array}$$

is a homomorphism of Lie algebras, i.e.

$$[X_H, X_G] = X_{\{H,G\}}.$$

Weihnachtsaufgabe 11: [Emmy Noether theorem]

In the setting of Weihnachtsaufgabe 10, for any $G, H \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ let us denote by ϕ_G^s and ϕ_H^t the flows (assumed global) of the ordinary differential equations

$$\dot{x} = X_G(x) \text{ and } \dot{x} = X_H(x), \text{ respectively.}$$

Show that:

(i) $G(\phi_H^t(x)) = \text{const.}$ for all $t \in \mathbb{R}$ if, and only if, $\{H, G\} = 0$.

(ii) $\phi_G^s \phi_H^t = \phi_H^t \phi_G^s$ for all $s, t \in \mathbb{R}$ if, and only if, $[X_H, X_G] = 0$.

(iii) $G(\phi_H^t(x)) = \text{const.}$ for all $t \in \mathbb{R}$ if, and only if, $\phi_G^s \phi_H^t = \phi_H^t \phi_G^s$ for all $s, t \in \mathbb{R}$.

Weihnachtsaufgabe 12: At the beginning of the 17th century, the brilliant astronomer Johannes Kepler used observational data by Tycho Brahe (and his daughters) to derive the laws of planetary motion. Thanks to the precision of his methods (some might even call them witchcraft), he was able to oppose well-believed and little checked "scientific facts". A man of faith, his results enabled him to trace a series of rare conjunctions of Jupiter and Saturn back, all the way, to at least 7 B.C. Such rare events may well have enlightened more than just three magi to seek the light.

In the same setting as 10, 11, with $N = 3$, suppose that the planetary 2-body motion is governed by the Hamiltonian $H = \frac{1}{2}||p||_2^2 + V(q)$ with the Newton potential $V(q) = 1/||q||_2$. Here p is interpreted as the velocity of a particle at position q. Kepler's first law states that the planets follow ellipses, with the Sun at one of the foci. Use what you have learned in the previous exercises to prove and interpret Kepler's second law:

The line joining a planet to the Sun sweeps equal areas in equal times.

Hint: Notice the invariance of the provided Hamiltonian under rotations in $\mathbb{R}^3$.