Bifurcation Theory

Note on norm continuity versus strong continuity

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We explore why norm continuity is too strong a concept for representations of Lie groups on Banach spaces X. Any Lie group contains one-parameter subgroups $t \mapsto \exp(at), t \in \mathbb{R}$, via elements a of its Lie algebra. Indeed $\exp(a(t+s)) = \exp(at)\exp(as)$. We therefore address linear representations ρ of the group $(\mathbb{R}, +)$ on X, i.e.

$$\rho(t+s) = \rho(t)\rho(s), \qquad \rho(0) = \mathrm{id}_X. \tag{1}$$

Slightly more generally, we will study *semigroups*, where $\rho(t)$ is only required to be defined for $t \geq 0$ and (1) is only required to hold for all $t, s \geq 0$.

Strong continuity only requires continuity of the map

$$\begin{aligned} [0,\infty) \times X &\to X \\ (t,x) &\mapsto \rho(t)x. \end{aligned} \tag{2}$$

In particular $\rho(t) \in L(X)$, i.e. $\rho(t)$ is bounded linear, for all $t \geq 0$. *Norm continuity* requires that

$$\begin{aligned} \rho : [0,\infty) &\to L(X) \\ t &\mapsto \rho(t) \end{aligned} \tag{3}$$

is continuous, in the much stronger norm

$$||A||_{L(X)} := \sup_{|x| \leq 1} |Ax| \tag{4}$$

on the Banach space $L(X)$. In this note we prove

Theorem *Let $\rho(t)$ be a linear semigroup in the sense of (1). Then ρ is norm continuous if, and only if, there exists a bounded linear operator $A \in L(X)$ such that for all $t \geq 0$ we have*

$$\rho(t) = \exp(At). \tag{5}$$

In particular, ρ is then analytic in t, and $x(t) := \rho(t)x_0$ satisfies the following ordinary differential equation in the Banach space X:

$$\begin{aligned} \dot{x}(t) &= Ax(t)\,, \\ x(0) &= x_0\,. \end{aligned} \tag{6}$$

Moreover, any $\rho(t)$ is invertible with bounded inverse $\rho(t)^{-1} = \exp(-At)$, and ρ extends uniquely to a norm continuous linear representation of the group $(\mathbb{R}, +)$.

Example 1. Consider the strongly continuous *shift representation*

$$(\rho(t)x)(\xi) := x(t+\xi), \tag{7}$$

of $(\mathbb{R}, +)$ on $x(.) \in L^2(\mathbb{R})$ or $BC^0(\mathbb{R})$. Then (6) would require

$$(Ax_0)(\xi) = \tfrac{\partial}{\partial t}\Big|_{t=0}\, x(t+\xi) = x'(\xi) \in X\,, \tag{8}$$

i.e. $x(.)$ is in the Sobolev space $H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$ of functions with square integrable weak derivative, or in the space $BC^1(\mathbb{R})$ of bounded functions $x(.)$ with bounded continous derivative. In other words, $u(t, \xi) := (\rho(t)x_0)(\xi)$ is a weak solution of the partial differential equation

$$\frac{\partial}{\partial t}u - \frac{\partial}{\partial \xi}u = 0 \tag{9}$$

with initial condition $u(t, \xi) = x_0(\xi)$ at $t = 0$. Alas, in virtue of theorem 1, the representation $\rho(t)$ cannot be norm continuous because $H^1 \subsetneqq L^2$ and $BC^1 \subsetneqq BC^0$ are proper subspaces.

Example 2. An interesting example of a semigroup $\rho(t)$, say on $x \in X := L^2(\mathbb{R})$, is given by the solutions of the heat equation

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u, \tag{10}$$

again with initial condition $u(t, \xi) = x_0(\xi)$ at $t = 0$. The explicit solution for $t > 0$ is given by convolution with the heat hernel:

$$u(t, \xi) = (\rho(t)x_0)(\xi) := \int_{-\infty}^{+\infty} \tfrac{1}{\sqrt{4\pi t}} \exp(-(\xi - \eta)^2)/(4t))x_0(\eta)d\eta\;, \tag{11}$$

thanks to Fourier transformation of (10). Note strong continuity of $\rho(t), t \geq 0$, with $\rho(0) = \text{id}$. Since $\rho(t)x_0 \in C^\infty(\mathbb{R})$ is smooth, for any $t > 0$, norm continuity of $\rho(t)$ must fail: the maps $\rho(t)$ are not even surjective, onto $X = L^2$, much less invertible.

Proof of theorem. To show necessity, suppose A is bounded linear. Then the majorant $\exp(||A||\,t)$ shows absolute convergence of the power series

$$\rho(t) := \exp(At) := \sum_{k=0}^{\infty} \tfrac{1}{k!}(At)^k \tag{12}$$

in $L(X)$. This also shows norm continuity, differentiability, analyticity, and that $\rho(t)$ is the flow of the ODE (6).

Sufficiency is more interesting. Let ρ be norm continuous and pick $h > 0$. (In a moment we will choose h small enough.) Define the "regularization"

$$R_h x := \tfrac{1}{h} \int_0^h \rho(s) x \, ds \tag{13}$$

Then

$$\begin{aligned} |R_h x - x| &= \left| \tfrac{1}{h} \int_0^h (\rho(s) - \mathrm{id}) x \, ds \right| \leq \tfrac{1}{h} \int_0^h ||\rho(s) - \mathrm{id}|| \cdot |x| \, ds \leq \\ &\leq \sup_{0 \leq s \leq h} ||\rho(s) - \mathrm{id}|| \cdot |x|. \end{aligned} \tag{14}$$

In other words, norm continuity (3), (4) implies

$$||R_h - \mathrm{id}|| \leq \varepsilon < 1 \ , \tag{15}$$

if we fix $h > 0$ small enough. The Neumann series (or contraction mapping) allows us to invert $R_h \in L(X)$:

$$R_h^{-1} = (\mathrm{id} - (\mathrm{id} - R_h))^{-1} := \sum_{k=0}^{\infty} (\mathrm{id} - R_h)^k \ , \tag{16}$$

with absolute convergence by the geometric series majorant $\sum ||\mathrm{id} - R_h||^k \leq \sum \varepsilon^k = 1/(1-\varepsilon)$. We may therefore define the bounded linear operator

$$A := \tfrac{1}{h}(\rho(h) - \mathrm{id}) R_h^{-1}. \tag{17}$$

Let $x_0 \in X, \ y := R_h^{-1} x_0$. Then (13) and the semigroup property (1) imply

$$\begin{aligned} x(t) := \rho(t) x_0 = \rho(t) R_h y &= \frac{1}{h} \int_0^h \rho(t) \rho(s) y \, ds = \\ = \frac{1}{h} \int_0^h \rho(t+s) y \, ds &= \frac{1}{h} \int_t^{t+h} \rho(s) y \, ds \, . \end{aligned} \tag{18}$$

Since the integral is continuous we may differentiate (18) with respect to t to obtain

$$\begin{aligned} \dot{x}(t) &= \frac{1}{h}(\rho(t+h) - \rho(t)) y = \\ &= \frac{1}{h}(\rho(h) - \mathrm{id}) \rho(t) R_h^{-1} x_0 = \\ &= \frac{1}{h}(\rho(h) - \mathrm{id}) R_h^{-1} \rho(t) x_0 = A x(t) \end{aligned} \tag{19}$$

and, of course, $x(0) = x_0$. Here we have used definition (17) of A. We have also used that $\rho(t)$ commutes with R_h, and therefore with R_h^{-1}. This proves the ODE claim (6) of the theorem.

Uniqueness of solutions of (6), for bounded linear $A \in L(X)$, follows by the usual Picard-Lindelöf iteration. That iteration, initialized with $x(t) \equiv x_0$, also provides the series $\rho(t) = \exp(At)$. This proves the theorem. $\bowtie$

Evidently, the construction of the infinitesimal generator A in (17) above does not really depend on the specific value of h. For $\rho(t) = \exp(At)$, as in (5), this can also be verified directly. Our construction, however, was based on the invertibility of R_h in (16) which, in turn, required some norm continuity.

The straightforward definition suggested by the differential equation (6),

$$Ax_0 := \lim_{h \searrow 0} \tfrac{1}{h} \left(\rho(h) x_0 - x_0 \right) \tag{20}$$

is considerably more interesting for only strongly continuous semigroups $\rho(t)$. Under an exponential growth assumption

$$||\rho(t)|| \leq M e^{\alpha t}\,, \tag{21}$$

for suitable constants $M, \alpha > 0$ and all $t \geq 0$, the *infinitesimal generator* A of $\rho(t)$ is then only defined for a dense subset of x_0 in X, but A is closed and satisfies a resolvent estimate.

Conversely, any densely defined, closed operator A with such a resolvent estimate generates a unique strongly continuous semigroup $\rho(t)$. Moreover, $\rho(t)$ satisfies the growth estimate (21). One definition of $\rho(t)$ proceeds via implicit Euler steps for the differential equations (6),

$$\rho(t) x_0 = \text{“}\exp\text{”}(At)\, x_0 := \lim_{n \to \infty} \left(\mathrm{id} - \frac{t}{n} A \right)^{-n} x_0\,. \tag{22}$$

See the books by [Pazy], [HiPhi], [Tanabe], [Henry], [Amann] for these results, and many more details, on semigroup theory for partial differential equations.

References

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