Homework Assignments

Dynamical Systems I

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 13: Solve the following initial-value problems by separation of variables and determine the maximal time intervals of existence of the solutions:

(i) $\dot{x} = x^2 e^t, \qquad x(0) = 1,$

(ii) $\dot{x} = 1 + x^2, \qquad x(0) = 0,$

(iii) $\dot{x} = 4 - x^2, \qquad x(0) = 0,$

(iv) $\dot{x} = x(2-x)^2, \qquad x(0) = 1,$

Problem 14: [Hai-Alarm am Müggelsee] Consider the closed, sealed-off Müggelsee with predator and prey fish of positive total masses x and y, respectively. Suppose their dynamics obeys the Volterra-Lotka equations

$$\begin{aligned} \dot{x} &= x(\mu - \nu y), \\ \dot{y} &= y(-\varrho + \sigma x), \end{aligned}$$

with positive fixed parameters $\mu, \nu, \varrho, \sigma$. ε-fishing ($0 < \varepsilon < \mu$) would change μ into $\tilde{\mu} = \mu - \varepsilon$ and ϱ into $\tilde{\varrho} = \varrho + \varepsilon$, with $\varepsilon > 0$. Why?

Does the time-averaged prey population

$$\overline{x} := \lim_{t \to \infty} \frac{1}{t} \int_0^t x(\tau)\, \mathrm{d}\tau$$

increase or decrease, due to fishing? What happens to the total population $\overline{x+y}$?

Hint: Consider time averages of $\dot{x}/x$, $\dot{y}/y$.

Problem 15: [First integrals] Consider

$$\begin{aligned} \dot{x} &= -y^3 \\ \dot{y} &= x^3, \end{aligned}$$

$x, y \in \mathbb{R}$. Using a first integral, prove that there exist periodic solutions of (not necessarily minimal) period 4. Show that these 4-periodic solutions satisfy the delay equation

$$\dot{x} = -x^3(t-1).$$

Problem 16: [Epidemics] Consider the following SIR model discussed in the lecture:

$$\begin{aligned} \dot{S} &= -\beta IS \\ \dot{I} &= \beta IS - \gamma I \\ \dot{R} &= \gamma I, \end{aligned}$$

where S denotes the percentage of susceptible people in a population, I the percentage of infected, and R the percentage of recovered (removed) people.

Prove, disprove, or criticize the following claims from the english Wikipedia site on compartmental models in epidemiology:

(i) for $t \geq 0$: $S(t) = S(0)\exp(-\beta/\gamma(R(t) - R(0)))$

(ii) $S_{end} = 1 - R_{end} = S(0)\exp(-\beta/\gamma(R_{end} - R(0)))$. "This shows that at the end of an epidemic, unless $S_0 = 0$, not all individuals of the population have been removed, so some must remain susceptible. This means that the end of an epidemic is caused by the decline in the number of infectious individuals rather than an absolute lack of susceptible subjects."

(iii) if $S(0) > \gamma/\beta$, then $\dot{I}(0) > 0$, "i.e., there will be a proper epidemic outbreak with an increase of the number of the infectious (which can reach a considerable fraction of the population)."

(iv) if $S(0) < \gamma/\beta$, then $\dot{I}(0) < 0$, "i.e., independently from the initial size of the susceptible population the disease can never cause a proper epidemic outbreak. As a consequence, it is clear that the basic reproduction number β/γ is extremely important."