Homework Assignments

Dynamical Systems I

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http://dynamics.mi.fu-berlin.de/lectures/

due date: Thursday, May 28, 2020, 23:59

All points of this sheet will be counted as Extra Credits, i.e., we will count them towards your total number of points, but not increase the number of points needed to pass the course.

Problem 17: Sketch the phase portraits of $\ddot{x} + V'(x) = 0$,

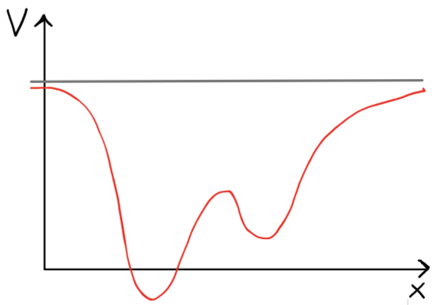
(i) for the Kepler problem,

$$V(x) = -\frac{1}{x} + C\frac{1}{x^2}, \qquad C > 0,\ x > 0,$$

(ii) for the Yukawa potential,

$$V(x) = -\frac{e^{-x}}{x}, \qquad x > 0,$$

(iii) for the following (red) potential,



Pay attention to saddle equilibria, homoclinic orbits, and asymptotic behavior at infinity.

Supplementary extra credit: explain why (i) is indeed the Kepler problem and describes the motion of a planet.

Problem 18: Consider the potential

$$V(x) = x^4,$$

and the associated pendulum equation

$$\ddot{x} = -V'(x).$$

Prove or disprove:

For every $p > 0$ there exists a periodic orbit with minimal period p such that $x(t) = x(-t)$, for all $t \in \mathbb{R}$.

Problem 19: Consider the pendulum equation

$$\ddot{x} + g(x) = 0$$

for a continuous, odd function $g : \mathbb{R} \to \mathbb{R}$, i.e. $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. Assume $g(x) \cdot x > 0$ for all $x \neq 0$. Let $p(g, a) > 0$ be the minimal period of the solution to the initial value $x(0) = a > 0$, $\dot{x}(0) = 0$.

Prove:

(i) If $g_1(x) < g_2(x)$ for all $x > 0$ then $p(g_1, a) > p(g_2, a)$ for all $a > 0$.

(ii) [Soft spring] If $x \mapsto g(x)/x$ is strictly monotonically increasing for $x > 0$, then $a \mapsto p(g, a)$ is strictly monotonically increasing for $a > 0$.

Hint: $y(t) := \frac{a_1}{a_2} x(t)$ solves the equation $\ddot{y} + \tilde{g}(y) = 0$ with $\tilde{g}(y) := \frac{a_1}{a_2} g(\frac{a_2}{a_1} y)$.

Problem 20: [The Lorenz attractor] The Lorenz equations are given by

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz. \end{aligned}$$

(i) Find all the equilibrium points of the Lorenz equations.

(ii) Fix $\sigma = 10$, $b = 8/3, r = 28$. Choose any non-equilibrium initial condition (x_0, y_0, z_0) and, using the numerical integrator of your choice, plot the solution trajectory up to times $T = 10, T = 100, T = 1000$. Now perturb your initial condition by $\varepsilon = 10^{-6}$ in an arbitrary direction and plot in the same plot as the solution starting at (x_0, y_0, z_0). Describe and discuss your observations.

Supplementary extra credit: Try different values of r. Which qualitatively different phenomena do you observe?