Problem 21: Use your favorite numerical integrator to plot the solution of the Van-der-Pol oscillator
\[\varepsilon \dot{x} = -y + x(1 - x^2)\]
\[\dot{y} = x\]
with initial conditions \(x(0) = 1, y(0) = 0\) up to time \(t = 10\). Choose parameters \(\varepsilon = 0.16, 0.01, 0.0025, 0.0009, \) and \(0.000625\). Use stepsize \(h = 10^{-3}\). Compare explicit Euler, Runge Kutta, and another solver of your choice. Describe and discuss observations and problems.

Extra credit: What happens for \(\varepsilon = 0.0009\) to the numerical solution calculated by the Runge-Kutta solver with fixed stepsize \(h = 10^{-3}\)?

Problem 22: Consider the synthesis of hydrogen iodide from hydrogen and iodine,

\[\text{H}_2 + \text{I}_2 \rightleftharpoons 2\text{HI},\]

is well described by mass-action kinetics. Denote the reaction rates by \(k_-, k_+\) and determine the kinetic equations for the vector \(x = (x_{H_2}, x_{I_2}, x_{HI})\) of concentrations. Determine the \(\omega\)-limit to initial data \(x^0 = (x^0_{H_2}, x^0_{I_2}, 0)\).

Extra credit: Which choice of initial data yields the highest gain?

Problem 23: Consider the initial-value problem
\[\dot{x} = f(x) := Ax, \quad x(0) = x_0,\]
\(A \in \mathbb{R}^{n \times n}\) on the time interval \(0 \leq t \leq T\) for fixed \(T > 0\). Calculate an approximate solution \(x(T)\) analytically by Picard iteration, i.e., determine the \(n\)-th Picard iterate \(x[n](T)\);
\[x_{[k+1]}(t) = x_0 + \int_0^t f(x[k](\tau)) \, d\tau, \quad x[0](t) \equiv x_0.\]
Now consider \(A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and \(x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\).

How fast do the Picard iterations approximate the exact solution?
**Problem 24:** The initial-value problem

\[ \dot{x} = f(x) := x^2, \quad x(0) = x_0 := 2 \]

has a solution for \(-\infty < t < \bar{\tau}\) with “blow-up”, \(\lim_{t \to \bar{\tau}} x(t) = +\infty\).

Let \((x_k)_{k \in \mathbb{N}}\) be the series of Picard iterates:

\[
\begin{align*}
    x_0(t) & \equiv x_0, \\
    x_{n+1}(t) & = x_0 + \int_0^t f(x_n(s)) \, ds.
\end{align*}
\]

(i) Prove: \(x_k(t)\) is defined for all \(k \in \mathbb{N}\) and \(t \in \mathbb{R}\).

(ii) Calculate \(x_1(t), x_2(t), x_3(t)\) and \(x_4(t)\) explicitly.

(iii) Determine all \(t \geq 0\) such that \(x_k(t)\) converges to the solution \(x(t)\) of the initial-value problem, as \(k \to \infty\).