Homework Assignments

Dynamical Systems I

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Problem 29: Consider the Banach space $Y := C^0([0,1],\mathbb{R})$ with the usual sup-norm.

(i) Show that

$$\begin{aligned}\mathcal{F} : Y &\to Y\\ u &\mapsto \Big(t \mapsto \big(u(t)\big)^2\Big).\end{aligned}$$

is continuously differentiable and its Fréchet derivative is given by

$$\Big(\big(\mathcal{F}'(u)\big)v\Big)(t) = 2u(t)\cdot v(t).$$

(ii) Show that

$$\begin{aligned}\mathcal{G} : Y &\to \mathbb{R}\\ u &\mapsto \left(t \mapsto \int_0^1 \big(u(t)\big)^2 dt\right).\end{aligned}$$

is continuously differentiable and determine its Fréchet derivative.

Problem 30: Consider the Banach space BC^1 of continuously differentiable vector fields $f:\ X \to X = \mathbb{R}^n$ with

$$\|f\|_{BC^1} := \sup_{x\in X}(|f(x)| + |f'(x)|) \ < \ \infty.$$

Let f, g be vector fields in BC^1 and $x(f,t)$ denote the solution at time t of the differential equation

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0.$$

Is the map

$$x(t,\cdot):\ BC^1 \to X, \qquad f \mapsto x(t,f),$$

differentiable with respect to $f \in BC^1$, for fixed t ? If so then which differential equation is solved by the variation $v(t) := \mathrm{D}_f x(t,f)g$?

Problem 31: Let $f : X \to X = \mathbb{R}^n$ be locally Lipschitz continuous and $J(x_0) = (t_-(x_0), t_+(x_0))$ the maximal interval of existence of the solution to the initial-value problem

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0.$$

Prove that the map $x_0 \mapsto t_+(x_0) \in (0, \infty]$ is lower semi-continuous. *Reminder:* A map g is called lower semi-continuous in x_0 if

$$\begin{aligned} &\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \quad (|x - x_0| < \delta \Rightarrow g(x) - g(x_0) > -\varepsilon)\,, && \text{in the case} \quad g(x_0) < \infty, \\ &\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \quad (|x - x_0| < \delta \Rightarrow g(x) > 1/\varepsilon)\,, && \text{in the case} \quad g(x_0) = \infty. \end{aligned}$$

Problem 32: Calculate the solutions of the following linear differential equations

$$\text{(i)} \qquad \dot{x} = \begin{pmatrix} -3 & 0 & 2 \\ 0 & 0 & 5 \\ -1 & 0 & 0 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\text{(ii)} \qquad \dot{x} = \begin{pmatrix} 4 & -2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$