Homework Assignments

Dynamical Systems I

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 33: We want to understand the damped linear pendulum

$$\ddot{x} + \nu \dot{x} + \omega_0^2 x = 0$$

with parameters ν, $\omega_0 > 0$ and initial conditions $x(0) = 0$, $\dot{x}(0) = 1$.

(i) Determine the explicit solution of the given initial-value problem for all ν, ω_0. Sketch phase portraits and a diagram of the (ν, ω_0)-plane of different qualitative behavior.

(ii) How does the phase portrait change at the boundary between different zones in the (ν, ω_0)-plane, for instance due to a change of the damping? How do you reconcile the discontinuities in the phase portraits of the Jordan normal form with differentiable dependence of the flow on the parameters ν, ω_0?

Problem 34: Consider the sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots,$$

i.e.,

$$x_n \ = \ \frac{1}{2^n \sqrt{5}} \left(\left(1 + \sqrt{5}\right)^n - \left(1 - \sqrt{5}\right)^n \right).$$

(i) Give an interpretation of this sequence via iterations of a suitable linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$. Determine the linear map and prove your claim.

(ii) For which initial conditions $(\tilde{x}_1, \tilde{x}_2) \in \mathbb{Z}^2$ to the above iteration does the quotient

$$r_n \ = \ \tilde{x}_{n+1} / \tilde{x}_n$$

converge to the “golden ratio” $g = \frac{1}{2}(1 + \sqrt{5})$?

Problem 35: [Lissajous figures] Let A be a symmetric real (2×2)-matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}.$$

Consider the Hamilton system with Hamilton function $H(x, \dot{x}) = \frac{1}{2}(\dot{x}^{\mathrm{T}}\dot{x} + x^{\mathrm{T}}Ax)$:

$$(*) \qquad \ddot{x} = -Ax.$$

(i) Transform $(*)$ into a system of decoupled pendulum equations (ω_1, ω_2 real):

$$(**) \qquad \begin{cases} \ddot{y}_1 + \omega_1^2 y_1 &= 0, \\ \ddot{y}_2 + \omega_2^2 y_2 &= 0, \end{cases}$$

(ii) Sketch the solution $(x_1(t),\ x_2(t))$ of $(*)$ for

$$A = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

with initial conditions $x_1 = x_2 = \dot{x}_1 = -\dot{x}_2 = 1$.

Problem 36: Let Φ_t be a flow on a metric space X. Let x_0 be "stable", i.e.

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in X \quad \Big(\ |x - x_0| < \delta \quad \Longrightarrow \quad \forall\, t \geq 0 \quad |\Phi_t(x) - \Phi_t(x_0)| < \varepsilon\ \Big)$$

Prove or disprove: x_0 is an equilibrium of the flow.