Problem 37: Consider the forced damped linear pendulum

\[ \ddot{x} + \nu \dot{x} + \omega_0^2 x = f \cos(\omega t) \]

with parameters \( f, \nu, \omega_0, \omega > 0 \) and initial conditions \( x(0) = 0, \dot{x}(0) = 1 \).

(i) Does there exist a unique periodic solution?
(ii) Calculate its amplitude \( A(\omega) \) and sketch it.
(iii) For which parameter combinations does there exist a unique global maximum \( A(\omega_{\text{max}}) \)? Show that \( \omega_{\text{max}} < \omega_0 \).

*Hint:* We call \( \omega_{\text{max}} \) the resonant frequency.

Problem 38: How many digits (in the decimal system) does the 202.020.202.020-th entry of the sequence \((1, 3, 8, 20, 48, 112, \ldots)\) have, i.e. \( x_n = 4x_{n-1} - 4x_{n-2} \) with \( x_0 = 1 \) and \( x_1 = 3 \)?

Problem 39: Consider the system of differential equations

\[ \dot{x}_i = x_i \left( (Ax)_i - x^T Ax \right), \quad i = 1, \ldots, N, \]

for \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \) with \( x_i \geq 0 \) and

\[ \sum_{i=1}^{N} x_i = 1. \]

Consider the case of the identity matrix, \( A = \text{id} \).

(i) Sketch the phase portraits for \( N = 2, 3, 4 \).
(ii) Describe the set of equilibria and the set of heteroclinic orbits for arbitrary \( N \). In particular determine which equilibria are connected by heteroclinic orbits.

*Extra credit:* What happens for \( \sum_{i=1}^{N} x_i \neq 1 \) and/or \( x_i < 0 \) for some \( i \)?

*Hint:* Enumerate equilibria \( x^* \) by subsets \( M(x^*) = \{ i | x^*_i \neq 0 \} \subseteq \{1, \ldots, N\} \).
Problem 40: Let $f$ be a differentiable vector field such that each trajectory is bounded.

Prove or disprove: The $\omega$-limit depends continuously on the initial condition, i.e. if

$$\lim_{n \to \infty} \text{dist}(x_n, x) = 0,$$

then

$$\lim_{n \to \infty} \text{dist}(\omega(x_n), \omega(x)) = 0.$$

Here we use the symmetric Hausdorff distance

$$\text{dist}(A, B) := \max \left( \sup_{a \in A} \inf_{b \in B} \text{dist}(a, b), \sup_{b \in B} \inf_{a \in A} \text{dist}(a, b) \right).$$