Dynamical Systems I

Bernold Fiedler

Häusliche Nachklausur – Second Take-home exam
November 3, 2020, 10-12

Please follow the guidelines.
Failure to do so will invalidate your exam.

1. Confirm, via email to dynamics.exercises@gmail.com by 10:30 a.m., that you received the exam.

2. Send your solutions to dynamics.exercises@gmail.com by 12:00 a.m., as a single .pdf file. Mark your name and Matrikelnummer clearly.

3. Include in your email, and thereby endorse, the pertinent legal statement "Selbständigkeitserklärung" sent to you on Monday.

All claims must be justified.

Problem 1 (1 point): What is the definition of an evolution $\Psi_{t,s}$ on $\mathbb{R}^N$? Which differential equation is associated to an evolution $\Psi_{t,s}$ and in what sense does the evolution "solve" the equation?

Problem 2 (1 point): Formulate the theorem of Picard and Lindelöf on local existence and uniqueness of the solution to the initial value problem of an autonomous ordinary differential equation on $\mathbb{R}^N$.

Problem 3 (1 point): How can the solution of the initial-value problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$, be written as the fixed point of a map in a Banach space? Specify the Banach space, its norm, and the map.

Problem 4 (1 point): Let a, b, u be continuous, non-negative functions of $t \geq 0$, such that

$$u(t) \ \leq \ a(t) + \int_0^t b(s)u(s)\,\mathrm{d}s,$$

for all $t \geq 0$. How can $u(t)$ be estimated in terms of a, b ?

Problem 5 (1 point): How is the local asymptotic stability of an equilibrium related to the linearization of the C^1-vector field?

Problem 6 (1 point): What is the explicit flow to the linear differential equation

$$\dot{x} = Ax,$$

with a constant matrix A, in complex Jordan normal form?

Problem 7 (1 point): Let the forward orbit $\gamma^+(x_0)$ of x_0 under a flow in $\mathbb{R}^N$ be bounded. Prove or disprove: the ω-limit set of x_0 is open.

Problem 8 (1 point): Formulate the theorem of Poincaré & Bendixson on ω-limit sets of bounded orbits of planar vector fields.

Problem 9 (4 points): Let Φ_t be a C^1-flow on a Banach space X. Assume that the orbit of x_0 possesses arbitrarily small periods. Show that x_0 is an equilibrium.

Problem 10 (4 points): Identify the ω-limit sets of <u>each</u> trajectory of

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \sin x_2 \\ \sin x_1 \end{pmatrix}, \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$