Homework Assignments

Dynamical Systems II

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http://dynamics.mi.fu-berlin.de/lectures/

due date: Thursday, December 3, 2020, 16:00

Problem 9: Let $0 < \beta < 1$ be irrational and

$$s_n := \operatorname{sign}(\cos(n\pi\beta)), \qquad n = 1, 2, 3, ...$$

The sequence

$$w_n := |s_n - s_{n+1}|/2 \in \{0, 1\}$$

detects sign changes of the sequence s_n. Prove that we can reconstruct β from the 01-sequence w_n as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} w_n = \beta.$$

Free extra: Is it possible to recover a rational number β from the sequence s_n?

Problems 10 and 11: Let A, B denote orientation-preserving $\mathcal{C}^2$-diffeomorphisms of S^1, AB composition, and ϱ the rotation number. Prove or disprove:

(i) $\varrho(A) + \varrho(A^{-1}) = 0$;

(ii) $\varrho(ABA^{-1}) + \varrho(B^{-1}) = 0$;

(iii) if $AB = BA$ then $\varrho(AB) = \varrho(A) + \varrho(B)$;

(iv) if $\varrho(A) = \varrho(B) \notin \mathbb{Q}$ then $AB = BA$;

(v) if $\varrho(A) = \varrho(B) \notin \mathbb{Q}$ then $\varrho(AB) = 2\varrho(A)$;

(vi) $\varrho(AB) = \varrho(BA)$;

(vii) if $n \in \mathbb{Z}$ then $\varrho(A^n) = n\varrho(A)$;

(viii) $\varrho(AB) = \varrho(A) + \varrho(B)$.

Problem 12: Consider the map $A(y) = 2y$ for $0 \leq y < 1$. Extend A to a map such that $A(y+1) = A(y) + 1$ for all y. Note the A defines a map $\tilde{A} : S^1 \to S^1 = \mathbb{R}/\mathbb{Z}$ given by $\tilde{A}(y) = 2y (\text{mod } 1)$. For any initial condition y_0 consider the usual "rotation number"

$$\varrho(y_0) = \lim_{k\to\infty} (A^k(y_0) - y_0)/k.$$

(i) Find periodic orbits of $\tilde{A}$. Which minimal periods $p > 0$ do occur?

(ii) Does $\varrho(y_0)$ depend on y_0? Compare your results to the theorem on rotation numbers proved in the lecture?