Homework Assignments

Dynamical Systems II

Bernold Fiedler, Alejandro López Nieto

`http://dynamics.mi.fu-berlin.de/lectures/`

due date: Thursday, December 17, 2020, 16:00

Problem 17: Consider the C^k vector field on $\mathbb{R}^N$, $k \geq 1$,

$$\dot{x} = f(x), \qquad f(0) = 0,$$

where 0 is a *hyperbolic* equilibrium. For the solution flow Φ_t and $T \in \mathbb{R}$ we define the stable and unstable sets

$$W_T^{\mathrm{s}}(0) := \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \Phi_{nT}x = 0,\ n \in \mathbb{N}\},$$

$$W^{\mathrm{s}}(0) := \{x \in \mathbb{R}^N \mid \lim_{t\to\infty} \Phi_t x = 0\},$$

$$W^{\mathrm{u}}(0) := \{x \in \mathbb{R}^N \mid \lim_{t\to-\infty} \Phi_t x = 0\}.$$

Prove or disprove:

(i) 0 is a hyperbolic fixed point under the iteration of Φ_T, $T \neq 0$.

(ii) $W_T^{\mathrm{s}}(0) \subset W^{\mathrm{s}}(0)$.

(iii) $W^{\mathrm{s}}(0) \subset W_T^{\mathrm{s}}(0)$ for any $T > 0$.

Problem 18: Notice that the global stable and unstable sets $W^{\mathrm{s}}(0)$, $W^{\mathrm{u}}(0)$, as defined in problem 17, do not require 0 to be a hyperbolic equilibrium. For the given vector fields discuss whether $W^{\mathrm{s}}(0)$ and $W^{\mathrm{u}}(0)$ are manifolds.

(a) $\begin{aligned}\dot{x} &= -x^2 y,\\ \dot{y} &= -xy^2.\end{aligned}$ (b) $\begin{aligned}\dot{x} &= x^2,\\ \dot{y} &= -y.\end{aligned}$ (c) $\dot{z} = z^2,\ z \in \mathbb{C}.$

(d) The vector field induced by (c) on the Riemann sphere $\hat{\mathbb{C}}$. In other words, (c) is an ODE on the coordinate chart $(\mathrm{Id}, \mathbb{C})$ of the Riemann sphere, use the biholomorphism (of $\hat{\mathbb{C}}$) $z \mapsto 1/z$ to extend the vector field to the whole Riemann sphere.

Problem 19: Consider the gradient vector field

$$\dot{x} \;=\; -\nabla V(x), \qquad x \in \mathbb{R}^N,$$

with C^2-Lyapunov function $V : \mathbb{R}^N \to \mathbb{R}$. Let $x = 0$ be a *hyperbolic* equilibrium.
Prove or disprove: Locally, the stable manifold satisfies

(i) $W^{\mathrm{s}}_{\mathrm{loc}}(0) \;\subseteq\; \{x \in \mathbb{R}^N \;;\; V(x) \geq V(0)\}_{\mathrm{loc}}$,

(ii) $W^{\mathrm{s}}_{\mathrm{loc}}(0) \;\supseteq\; \{x \in \mathbb{R}^N \;;\; V(x) \geq V(0)\}_{\mathrm{loc}}$.

Problem 20: Consider the pendulum

$$\ddot{x} + 3x^2 - 2x \;=\; 0.$$

Let

$$W^{\mathrm{s}}_{\mathrm{loc}}(0) \;=\; \{(x, \dot{x}) = (x, h(x)) \;;\; -\varepsilon < x < +\varepsilon\}$$

be the local stable manifold at the hyperbolic equilibrium $x = \dot{x} = 0$.
Determine the Taylor expansion

$$h(x) \;=\; \sum_{k=0}^{M} h_k x^k + \mathcal{O}(x^{M+1})$$

up to order $M = 3$.

Hint: Use the invariance of $W^{\mathrm{s}}_{\mathrm{loc}}$.