Homework Assignments

Dynamical Systems II

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 25: Given the iteration of a homeomorphism

$$x_{n+1} = \Phi(x_n),\ x_n \in X$$

let us explore the existence of shift dynamics on a compact invariant subset $I \subset X$, i.e. a subset on which

$$\Phi|_I = \tau \circ \sigma \circ \tau^{-1} \text{ for some homeomorphism } \tau : S \to I.$$

Discuss if shift dynamics possible or not for:

(i) $X = \mathbb{R}/\mathbb{Z}$.

(ii) X a real 2-dimensional manifold.

(iii) $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ the time-1 map of an autonomous C^1-vector field.

Problem 26: Let Φ be a homeomorphism containing shift dynamics on an invariant set I. Let $\tau : S \to I$ denote the homeomorphism which conjugates $\Phi|_I$ to the shift σ. As in class, we denote by $V(s) = \bigcap_{n=0}^{\infty} V_{s_0 \dots s_n}$ the local stable leaf of $p := \tau(s)$ for any element $s \in S$.

(i) Describe the global stable set $W^{\mathrm{s}}(p)$ of p, i.e. the set of q for which

$$\Phi^k(q) \in \bigcup_a V_a \text{ for all } k \geq 0, \text{ and } \lim_{k \to +\infty} \operatorname{dist}(\Phi^k(q), \Phi^k(p)) = 0.$$

(ii) Let V be defined as $V := \bigcup_{s' \in S_N} V(s')$, show that V is compact.

(iii) Show that I attracts V, i.e. for all $\varepsilon > 0$ there exists n_0 such that

$$n \geq n_0 \Rightarrow \operatorname{dist}(\Phi^n(V), I) < \varepsilon.$$

(iv) Is $W^{\mathrm{s}}(p)$ dense in the stable set V of I?

Problem 27: Can an autonomous ODE flow, i.e. a flow solving $\dot{x} = f(x)$, possess transverse homoclinic points to:

(i) Equilibria.

(ii) Periodic orbits.

In particular, use the reduction to a Poincaré map, to develop an appropriate notion of stable and unstable manifolds of periodic orbits.

Problem 28: Consider the Hénon map

$$\begin{aligned} \bar{x} &= 1 - \alpha x^2 + y, \\ \bar{y} &= \beta x. \end{aligned}$$

Find a horseshoe for parameters $\alpha \gg 1$ and $0 < \beta \ll 1$.

Hint: Choose a small $\varepsilon > 0$ and map the square $Q = [-\varepsilon, \varepsilon]^2$.

Extra: Detect a transverse homoclinic point.