Homework Assignments

Dynamical Systems II

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 37: Consider the vector field

$$\dot{x} = f(x),\ x \in \mathbb{R}^N,\ f(0) = 0.$$

Assume that the linearization $Df(0)$ possesses an algebraically simple eigenvalue 0 and all other eigenvalues have nonzero real part. Can there exist nonstationary periodic orbits in arbitrarily small neighborhoods of $x = 0$?

Problem 38: Consider the system of differential equations

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= -y + x^3. \end{aligned}$$

Use a (local) center manifold to decide whether the equilibrium $(x, y) = (0, 0)$ is asymptotically stable.

Hint: Use the invariance of the center manifold to calculate the necessary terms of its Taylor expansion.

Problem 39: [Vanderbauwhede] Consider the analytic ODE

$$\begin{aligned}\dot{x} &= \mu x - x^2, \\ \dot{y} &= y - x^2. \\ \dot{\mu} &= 0\end{aligned}$$

Suppose the local center manifold, $W^{\mathrm{c}}_{\mathrm{loc}}(0)$, is analytic and write it as a graph

$$y = \Psi(x, \mu) = \sum_{k=0}^{\infty} a_k(\mu) x^k.$$

(i) For small enough $\mu = 1/m$, $m \in \mathbb{N}$, derive the recursion

$$\left(\frac{k}{m} - 1\right) a_k = (k-1) a_{k-1}.$$

(ii) Conclude that $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ cannot possibly be analytic.

Extra credit: Can $W^{\mathrm{c}}_{\mathrm{loc}}(0)$ be C^∞?

Problem 40: Given the system

$$\begin{aligned}\dot{x} &= x^2, \\ \dot{y} &= -y,\end{aligned}$$

consider the function

$$h_\alpha(x) = \begin{cases} \alpha \exp(1/x), & x < 0, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Show that the graphs

$$M_\alpha = \{(x, h_\alpha(x)) \mid x \in \mathbb{R}\}, \ \alpha \in \mathbb{R}$$

are invariant under the given ODE.

(ii) Consider now a smooth cut-off function such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Given the cut-off system

$$\begin{aligned}\dot{x} &= \chi(x/\varepsilon) x^2, \\ \dot{y} &= -y,\end{aligned}$$

for $\varepsilon > 0$ small enough there exists a unique global center manifold which we denote by W^c_χ. In class we saw that, locally near zero, $W^c_\chi = M_\alpha$ for a suitable α. Discuss the dependence of α on the choice of the cut-off function χ.