Homework Assignments

Dynamical Systems II

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`http://dynamics.mi.fu-berlin.de/lectures/`

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Only attempted exercises will be discussed. The exercises are for extra credit, i.e. points you obtain will count towards your "Übungsschein", however we will not use these questions to calculate the threshold necessary to obtain the "Übungsschein".

Weihnachtsaufgabe 1: Little Dyna Mix is bored in quarantine. Looking out of the window she sees the two swings in the backyard. Covered in snow, one has a considerably longer chain than the other. She knows very well that when she uses one of them she swings back and forth in a regular way, but she has also noticed that it takes longer to loop once in the swing with the longer chain. Boredom brings ridiculous ideas to Dyna's mind and she begins to wonder what would happen if she switched swings every so often, while preserving angle and momentum.

Dyna is still too young to solve differential equations, but she knows an expert. Play the role of a sensible adult and tell Dyna why simulating her mental experiment in the real world is not safe for a 7 year old.

To do this consider the linear nonautonomous harmonic oscillator

$$\ddot{x} = -\omega(t)^2 x,\ x \in \mathbb{R},$$

where $\omega(t)$ is π-periodic with

$$\omega(t) := \begin{cases} 1, & 0 \le t < \pi/2, \\ 5, & \pi/2 \le t < \pi. \end{cases}$$

Discuss the α- and ω-limit sets of the solutions with initial conditions $(x(0), \dot{x}(0)) = (1, 0)$ and $(x(\pi/2), \dot{x}(\pi/2)) = (1, 0)$.

Hint: The period p of the solutions of the harmonic oscillator with constant frequency ω satisfies $p = 2\pi/\omega$.

Weihnachtsaufgabe 2: Consider the matrix ODE

$$\dot{X} = [A(t), X],$$

where $A(t),\ X(t) \in \mathbb{R}^{N\times N}$ are continuous and $[A, B] := AB - BA$.
Show that the eigenvalues of $X(t)$ are first integrals, i.e. they are independent of t.

Hint: Conjugate $X(t)$ by the Wronskian (or fundamental matrix) of $\dot{y}(t) = A(t)y(t)$.

Weihnachtsaufgabe 3: On $u = (p, q) \in \mathbb{R} \times \mathbb{R}$ consider the symplectic form $\omega[u, v] = u^T Jv$, given by the 2×2 matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A diffeomorphism Φ is *symplectic*, if the linearization $D\Phi(x)$ preserves the form ω, i.e. if

$$\omega[D\Phi(x)u, D\Phi(x)v] = \omega[u, v]$$

holds for all $x, u, v \in \mathbb{R}^2$. Let Φ_t denote the flow of the C^1 vector field $\dot{x} = f(x)$. Recall that the vector field $f = f(p, q)$ is *Hamiltonian*, if $f = J(D_p H, D_q H)^T$ for a scalar C^2 Hamiltonian function $H = H(p, q)$.

Prove that the flow Φ_t is symplectic if, and only if, the vector field f is Hamiltonian.

Weihnachtsaufgabe 4: Let $\tilde{A} : S^1 \to S^1 = \mathbb{R}/\mathbb{Z}$ be an orientation preserving homeomorphism. We say that $A : \mathbb{R} \to \mathbb{R}$ is a *lift* of $\tilde{A}$, if

$$A(x) - \tilde{A}(x \pmod 1) = k \in \mathbb{Z}, \text{ for all } x \in \mathbb{R}.$$

Prove or disprove:

(i) Every orientation preserving homeomorphism of the circle $\tilde{A}$ has a lift A.

(ii) If A is a lift of $\tilde{A}$, then it is unique.

(iii) If $\tilde{A}$ has two different lifts $A_1,\ A_2$, then

$$\lim_{j \to \infty} \frac{A_1^j(0)}{j} = \lim_{j \to \infty} \frac{A_2^j(0)}{j}$$

Weihnachtsaufgabe 5: Let $\tilde{A}$ be a C^2 circle diffeomorphism with irrational rotation number (of a lift A). Consider the set $\mathcal{H}$ of circle homeomorphisms which conjugate $\tilde{A}$ to a rigid rotation. Show that $\mathcal{H}$, in the sup-norm distance, is homeomorphic to a circle.

Weihnachtsaufgabe 6: Prove or disprove: An orientation preserving homeomorphism $\tilde{A} : S^1 \to S^1$ with *rational* rotation number (of a lift A) is topologically conjugate to a rigid rotation if, and only if, there exists $q \in \mathbb{N}$ such that $\tilde{A}^q = \mathrm{Id}$.

Weihnachtsaufgabe 7: Consider the general pendulum equation

$$\ddot{x} + \nabla V(x) \;=\; 0, \qquad x \in \mathbb{R}^N,$$

with potential $V : \mathbb{R}^N \to \mathbb{R}$. Let $x = \dot{x} = 0$ be a *hyperbolic* equilibrium.
Consider the Hamiltonian $H(x, \dot{x}) = \frac{1}{2}\dot{x}^T\dot{x} + V(x)$. Prove or disprove: Locally, the level set of the equilibrium is the union of its stable and unstable manifolds, i.e.

$$W^{\mathrm{u}}_{\mathrm{loc}}(0,0) \cup W^{\mathrm{s}}_{\mathrm{loc}}(0,0) \;=\; \{(x,\dot{x})\;;\; H(x,\dot{x}) = H(0,0)\}_{\mathrm{loc}}$$

(i) for one degree of freedom, i.e. $N = 1$;

(ii) for more degrees of freedom, i.e. $N > 1$.

Weihnachtsaufgabe 8: A matrix $A \in \mathrm{SL}_2(\mathbb{Z})$, induces an iteration on the torus via

$$x_{n+1} = \tilde{A}x_n, \qquad x_n \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2.$$

Assume $|\mathrm{trace(A)}| > 2$.
Prove or disprove:

(i) The fixed point 0 of $\tilde{A}$ is hyperbolic.

(ii) The global stable manifold $W^{\mathrm{s}}(0)$ is dense in $\mathbb{T}^2$.

Weihnachtsaufgabe 9: Consider the C^k vector field, $k \geq 1$,

$$\dot{x} = f(x), \; x \in \mathbb{R}^N,$$

with solution flow Φ_t and a hyperbolic equilibrium at 0. Show that

$$\bigcup_{t \geq 0} \Phi_t(W^{\mathrm{u}}_{\mathrm{loc}}(0)) = \{x_0 \in \mathbb{R}^N \mid \alpha(x_0) = 0\}$$

Given $2 \leq N \in \mathbb{N}$, consider the shift S space on N symbols, i.e.

$$S := \{s = (s_k)_{k\in\mathbb{Z}} \mid s_k \in \{1, \dots, N\}\}.$$

Equip S with the compact-open topology induced by the neighborhood basis

$$\mathcal{N}_j(s) := \{s' \in S \mid s_k = s'_k, \text{ for all } |k| \leq j\}.$$

Equivalently, consider the metric

$$\mathrm{d}(s', s) = \sum_{k\in\mathbb{Z}} \frac{1}{2^{|k|}} |s'_k - s_k|.$$

The following exercises will show that S is a *Cantor set* in the given topology.

Weihnachtsaufgabe 10: Prove that S is *totally disconnected*, i.e. the connected components are singletons.

Weihnachtsaufgabe 11: Prove that S is *perfect*, i.e. $s \in \mathrm{clos}(S \setminus \{s\})$ for any $s \in S$.

Weihnachtsaufgabe 12: Prove that S is *compact*, i.e. any sequence in S possesses a convergent subsequence.