Basic Questions of Dynamical Systems II

1. What is the definition of a flow Φ_t on $\mathbb{R}^N$?

2. What is the definition of an evolution $\Psi_{t,s}$ on $\mathbb{R}^N$?

3. What is the differential equation solved by the flow $\Phi_t \in C^1(\mathbb{R} \times \mathbb{R}^N)$?

4. Given a flow Φ_t and an initial condition $x_0 \in \mathbb{R}^N$, define the α- and ω-limit sets $\alpha(x_0)$ and $\omega(x_0)$.

5. Given a flow Φ_t, when do we say that an orbit $\gamma(x_0) := \{\Phi_t(x_0) \mid t \in \mathbb{R}\}$ is:

   (a) stationary.

   (b) periodic.

   (c) heteroclinic.

   (d) homoclinic.

6. What is the stroboscope map of a time periodic non-autonomous vector field $\dot{x} = f(t, x)$?

7. State the Poincaré-Bendixson theorem for autonomous vector fields on $\mathbb{R}^2$.

8. Consider the C^1-flow Φ_t solving the ODE $\dot{x} = f(x)$, $f(0) = 0$. What is the linearized ODE at 0? How is the flow of the linearized equation related to Φ_t?

9. Let $W(t, s)$ denote the Wronskian of the linear nonautonomous ODE

   $$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^N.$$

   How are $\det(W(t, s))$ and $\mathrm{trace}(A(\tau))$ related?

10. Consider an orientation preserving homeomorphism $A : \mathbb{R} \to \mathbb{R}$ with $A(x + 1) = A(x) + 1$, for all x. In which sense does A induce a homeomorphism $\tilde{A}$ on the circle? Define the rotation number of A.

11. How are existence and minimal periods of periodic points related to the rotation number of an orientation preserving homeomorphism $\tilde{A} : S^1 \to S^1$? Prove your claim.

12. Does it make sense to consider the rotation number for noninvertible, continuous circle maps?

13. Formulate an ergodic theorem for parallel flows on the 2-torus.

14. Formulate the Denjoy theorem for C^2-diffeomorphisms $\tilde{A} : S^1 \to S^1$.

15. When do we say that, in a Diophantine sense, an irrational number ρ is badly approximable by rationals? How do such irrationals allow us to improve the Denjoy theorem?

16. What is a *devil's staircase* for the rotation number of a parametric family of circle diffeomorphisms $\tilde{A}_\lambda : S^1 \to S^1$, $\lambda \in \mathbb{R}$?

17. How are local/global stable and unstable manifolds of hyperbolic equilibria of autonomous C^1 vector fields defined? How do they relate to the local/global stable and unstable manifolds of the time-1 map of the C^1 flow?

18. Formulate the theorem on the existence of local stable and unstable manifolds for hyperbolic equilibria of C^1 vector fields.

19. Formulate the theorem on the existence of local stable and unstable manifolds for hyperbolic fixed points of diffeomorphisms.

20. Are local/global stable and unstable manifolds of hyperbolic equilibria unique? What are the tangent spaces to stable and unstable manifolds at the equilibrium?

21. What is the Bernoulli shift on N symbols? Define the shift space, its metric, and the shift map.

22. How can we construct

    (a) periodic orbits of every period

    (b) a dense set of periodic orbits

    (c) a dense orbit

    for the shift on 2 symbols?

23. How does the shift on 2 symbols illustrate the butterfly effect?

24. Formulate the C^0 theorem on Smale's horseshoe.

25. Formulate the C^1 theorem on Smale's horseshoe. What do we mean by the required invariant cone families?

26. Sketch a horseshoe construction for the bouncing-ball map

$$\begin{aligned}\varphi_{k+1} &= \varphi_k + v_k,\\ v_{k+1} &= v_k - \gamma\cos(\varphi_k + v_k),\end{aligned}$$

under a suitable assumption on γ.

27. What are transverse homoclinic points of diffeomorphisms?

28. Formulate the λ-lemma.

29. How does a transverse homoclinic point give rise to shift dynamics? Only sketch and label the relevant picture.

30. Provide an example of a diffeomorphism possessing transverse homoclinics.

31. What is the Plykin attractor?

32. What is a strange attractor? Sketch an example and list relevant properties.

33. Define hyperbolic structure for diffeomorphisms on compact manifolds.

34. Can every compact C^1 surface host a diffeomorphism with a hyperbolic structure?

35. Give an example of a diffeomorphism of a compact manifold with a hyperbolic structure.

36. How is structural stability of a diffeomorphism defined?

37. State Anosov's theorem on structural stability of linear diffeomorphisms of the 2-torus.

38. Give at least two examples of structurally stable diffeomorphisms of the 2-torus.

39. Draw an example of a diffeomorphism of the 2-torus, which is not structurally stable.

40. Is the set of structurally stable diffeomorphisms of a compact manifold open in the C^1-topology?

41. Is the set of structurally stable diffeomorphisms of a compact manifold dense in the C^1-topology?