

Basic Questions of Dynamical Systems 3: Bifurcation Theory

Note: All vector fields and diffeomorphisms are assumed smooth.

All group representations are assumed strongly continuous.

1. Consider two vector fields $f, g \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and a matrix $A \in \mathbb{R}^{N \times N}$. How is the Lie derivative of f in direction g defined? How is the Lie bracket of f and g defined? What is the adjoint operator $\text{ad}A$?
2. Formulate the theorem on the normal form of a vector field near an equilibrium via the transpose of the linearization.
3. Given a vector field f with linearization A at an equilibrium, assume that $\text{ad}(A^T)$ has trivial kernel $\{0\}$ when restricted to the space of homogeneous polynomials of any degree $m \geq 2$. What is the normal form g of f at the equilibrium? How are the dynamics of f and g related?
4. Consider a vector field f with linearization A at an equilibrium. Assume $AA^T = A^TA$. Which additional symmetry does the $\text{ad}(A^T)$ normal form of f possess, to any finite order?
5. Consider a vector field f with real diagonal linearization A at an equilibrium. When do resonances occur? How are they related to the normal form of f ?
6. Sketch the bifurcation diagram for the truncation to order 3 of the Hopf normal form in the plane.
7. What is the Bogdanov-Takens bifurcation in the plane? Describe some of its most relevant qualitative characteristics.
8. Formulate a theorem on the normal form of a smooth local diffeomorphism $\Phi(x) = Bx + \dots$ near the fixed point $x = 0$, via $B^T[\cdot]_m$. Explain your notation.
9. What is a period doubling bifurcation? Sketch a bifurcation diagram for the normal form.
10. What is the symmetry group for the normal form of a torus bifurcation?
11. What is a Fredholm operator? What is the Fredholm index?

12. Describe the procedure of Lyapunov-Schmidt reduction for solving an equation $F(x) = 0$.
13. Formulate the finite-dimensional theorem by Crandall and Rabinowitz on stationary bifurcation from simple eigenvalues.
14. Give an explicit example of a supercritical and a subcritical pitchfork bifurcation and sketch the possible local bifurcation diagrams, including their (in-)stability.
15. Formulate the buckling problem of the Euler beam

$$\ddot{v}(t) + \lambda \sin(v(t)) = 0, \quad \dot{v}(0) = \dot{v}(1) = 0,$$

as a stationary bifurcation problem, in the setting of Crandall and Rabinowitz.

16. What are the main advantages of the Lyapunov-Schmidt reduction for ODEs compared to normal form theory? What are the main disadvantages?
17. What is a representation of a group Γ on a Banach space X ?
18. What is the canonical representation of a finite group Γ on a Banach space $\mathbb{R}^{|\Gamma|}$?
19. What is a strongly continuous representation of a group Γ on a Banach space X ?
20. Let Γ be a group with subgroup K and let ρ be a representation of Γ on a Banach space X . Define the set X_K of K -fixed vectors. Define the isotropy Γ_x of an element $x \in X$. Show Γ_x is closed and X_K is a Banach space.
21. Let Γ be a group with representation ρ on a Banach space X . How are the isotropy groups of elements on the same group orbit Γx related to each other? What is an isotropy subgroup of Γ ?
22. Given a representation ρ of a group Γ on a Banach space X . How is the isotropy lattice of Γ on X defined?
23. When do we say that a map $F : X \rightarrow X$ is Γ -equivariant? When do we say that a subspace $V \leq X$ is Γ -invariant?
24. When do we say that a group representation is irreducible?
25. What is the isotypic decomposition of a representation ρ of a group Γ on X ?
26. Formulate Schur's lemma. When is a representation of a group called absolutely irreducible?

27. Consider an irreducible complex representation of an Abelian group on a finite-dimensional complex vector space $X \neq \{0\}$. What is the complex dimension of X ? Prove your answer
28. When do we say that a group representation ρ on a Hilbert space $(X, \langle \cdot, \cdot \rangle)$ is orthogonal?
29. What are the irreducible representations of $SO(2)$ acting on $L^2(S^1, \mathbb{C})$ via the usual shift representation?
30. What are the spherical harmonics and how are they connected to the irreducible representations of $SO(3)$ on $\mathbb{R}^3$?
31. Under which conditions is a Lyapunov-Schmidt reduced equation equivariant under a group Γ ?
32. Formulate Vanderbauwhede's equivariant branching lemma.
33. What is a maximal isotropy subgroup for a representation of a group Γ on X ?