Basic Questions of
Bifurcations: Theory and Applications

Note: All vector fields are assumed smooth.

1. Formulate the Hopf bifurcation theorem and sketch a possible local bifurcation diagram.

2. Formulate the bifurcation of periodic orbits, in the Hopf setting, as a stationary $SO(2)$-equivariant bifurcation problem in suitable Banach spaces of periodic functions. Describe the action of SO(2).

3. What is a reversible vector field? Give one example, each, where reversibility holds and where reversibility fails.

4. How does reversibility affect the Hopf bifurcation scenario? What is the equivariance group of the resulting stationary bifurcation problem in suitable Banach spaces of periodic functions? Describe the group action.

5. Formulate the theorem on reversible Hopf bifurcation.

6. Reformulate the theorem on reversible Hopf bifurcation as a theorem on standard bifurcation from simple eigenvalues, for points fixed under a reversibility.

7. Formulate the theorem on local bifurcation of subharmonic solutions, in two real parameters.

8. Under which sufficient conditions do invariant tori bifurcarte in a subharmonic bifurcation scenario?

9. Given a nonstationary periodic solution of an ODE, what is its (spatio-temporal) symmetry?

10. When do we say that a nonstationary periodic solution is a rotating wave? When do we call it a discrete wave?

11. Formulate the theorem on equivariant Hopf bifurcation.

12. Sketch a proof of the theorem on equivariant Hopf bifurcation. (You may assume that previously covered material holds.)

13. What is the adjoint variational equation of a linear system $\dot{y} = A(t)y$ in $\mathbb{R}^N$?

14. How are solutions of $\dot{y} = A(t)y$ and its adjoint variational equation related?

15. When do we call a function $\psi : \mathbb{R} \to \mathbb{R}^N$ weakly differentiable in L^2?

16. In which sense does the Sobolev space $H^1(\mathbb{R}, \mathbb{R}^N)$ embeds into $BC^{0,1/2}(\mathbb{R}, \mathbb{R}^N)$? How do "boundary conditions" at infinity appear naturally?

Consider the following smooth ODE in $\mathbb{R}^N$ with 2π-periodic forcing

$$(1) \quad \dot{x} = f(x) + g(\lambda, t + \beta, x), \; g(0, t, x) = 0, \; g(\lambda, t, 0) = 0,$$

possessing a homoclinic solution $\Gamma(t) \xrightarrow{t \to \pm\infty} 0$ at parameter value $\lambda = 0$ and satisfying the nondegeneracy assumptions of the lecture.

17. What is the Melnikov function $M(\beta)$ of $\Gamma(t)$ in (1)?

18. What can we say about the homoclinic solution $\Gamma(t)$ in (1) for small changes of the parameter λ if the Melnikov function $M(\beta)$ possesses a zero?

19. For small parameter λ, assume that the Melnikov function $M(\beta)$ of $\Gamma(t)$ in (1) possesses a simple zero. Does the stroboscope map of the ODE (1) possess a transverse homoclinic solution to 0?

20. How can we use the Hamiltonian structure of an ODE to simplify the expression of the Melnikov function?

21. What does the Melnikov function tell us about the Bogdanov-Takens unfolding

$$\ddot{x} = \mu_1 + x^2 + \varepsilon(\mu_2 + x)\dot{x},$$

for small $\varepsilon > 0$.