Homework Assignments

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 5: Given a smooth ODE

$$\dot{x} = f(x) = Ax + \mathcal{O}(|x|^2),$$

we saw in the lecture that there exists a smooth diffeomorphism $\psi(x) = x + \mathcal{O}(|x|^2)$ such that for $g = \psi^* f$ the m-th Taylor term satisfies

$$\widetilde{T}_m g \in W_m = \ker(\,\mathrm{ad}_m(A^T)).$$

Prove or disprove: The choice of $\widetilde{T}_m \psi$ is unique.

[Extra credit] Prove or disprove: The choice of $\widetilde{T}_m g \in W_m$ is unique.

Problem 6: Consider the smooth parameter dependent ODE

$$(1)\ \dot{x} = f(\lambda, x),\ f(\lambda, 0) \equiv 0,\ (x, \lambda) \in \mathbb{R}^N \times \mathbb{R}.$$

We denote $A := D_x f(0,0)$ and we denote by $\widetilde{A}$ the linearization at $(0,0)$ of the extended system

$$(2)\ \begin{array}{l} \dot{x} = f(\lambda, x), \\ \dot{\lambda} = 0, \end{array} \qquad \text{i.e.} \quad \widetilde{A} := \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right).$$

Show that

$$p(x) \in W_m := \ker(\mathrm{ad}_m(A^T)) \Leftrightarrow \begin{pmatrix} \lambda^k p(x) \\ 0 \end{pmatrix} \in \widetilde{W}_{m+k} := \ker(\mathrm{ad}_{m+k}(\widetilde{A}^T)) \text{ for all } k \in \mathbb{N}.$$

Prove that we can obtain the normal form of the parameter dependent system (1) by taking the normal form of

$$(3)\ \dot{x} = f(0, x)$$

and replacing the coefficients by suitable polynomials in λ.

Problem 7: Let Φ_t^g denote the flow $\Phi_t^g(x_0) = x(t)$ of the ODE $\dot{x} = g(x)$, $x(0) = x_0$ on $\mathbb{R}^N$. Let γ denote any invertible real $N \times N$ matrix.

(a) For all t, g, γ, show the conjugation formula

$$\Phi_t^{\gamma(g \circ \gamma^{-1})} = \gamma(\Phi_t \circ \gamma^{-1}).$$

(b) Show that g is γ-equivariant, i.e. $\gamma g = g \circ \gamma$, if and only if the following statement holds

$$x(t) \text{ solves } \dot{x} = g(x), \text{ if and only if } \gamma x(t) \text{ does.}$$

Problem 8: Consider the real ordinary differential equation

$$\dot{x} = \lambda_1 + \lambda_2 x + x^3.$$

Determine the number of equilibria and their stability, depending on the values of the real parameters λ_1 and λ_2. Sketch the phase portraits close to the origin for each relevant region in the (λ_1, λ_2)-plane.