Homework Assignments

Dynamical Systems III

Bernold Fiedler, Alejandro López Nieto

http://dynamics.mi.fu-berlin.de/lectures/

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Problem 9: What does normal form theory tell us about the dynamics near 0 of a smooth vector field

$$\dot{x} = f(x) = \mathcal{O}(|x|^2), \qquad x \in \mathbb{R}^2.$$

Problem 10: Consider the smooth iteration on $\mathbb{R}^2$

$$x_{n+1} = F(x_n) = Bx_n + \mathcal{O}(|x_n|^2) \coloneqq \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x + \mathcal{O}(|x_n|^2).$$

We denote $W_m \coloneqq \mathrm{Ker}(B^T[\cdot]_m)$, where $B^T[\cdot]_m$ is the restriction to the space of homogeneous polynomials H_m of the *commutator*

$$B^T[\phi](x) \coloneqq B^T\phi(x) - \phi(B^T x),\ \phi \in C^\infty(\mathbb{R}^2, \mathbb{R}^2).$$

(i) Derive necessary and sufficient conditions on $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ under which there exists $m \geq 2$ such that $W_m \neq 0$.

(ii) Interpret your result: under which conditions on (λ_1, λ_2) and in which precise sense, can the iteration F be *linearized* smoothly?

Problem 11: Consider the smooth iteration on $\mathbb{R}$

$$x_{n+1} = F(x_n) = -x_n + a_2 x_n^2 + a_3 x_n^3 + \dots \qquad x \in \mathbb{R}.$$

We saw in the lecture that the normal form is odd, i.e. given by

$$y_{n+1} = -y_n + b_3 y_n^3 + b_5 y_n^5 + \dots.$$

Determine the stability of the fixed point $x = 0$, depending on a_2, a_3.

Problem 12: In the lecture we saw derived the following unfolding of the Bogdanov-Takens bifurcation

$$(1)\ u'' = \varepsilon(\mu_2 + u)u' + \mu_1 + u^2.$$

(i) Show that if $\mu_1 < 0$ we can rescale (1) so that at $\varepsilon = 0$ the resulting equation is

$$(2)\ \ddot{v} = v^2 - 1.$$

(ii) Sketch the phase plane of the Hamiltonian equation (2).

(iii) Show that the homoclinic solution of (2) is given explicitly by

$$v(t) = 3\tanh^2(t/\sqrt{2}) - 2.$$