Homework Assignments

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 13: Let $x^*(t)$ be a periodic solution with minimal period $T > 0$ of the smooth ODE

$$\dot{x} = f(x),\ x \in \mathbb{R}^N.$$

Let $\mathcal{P}$ denote the smooth Poincaré map defined locally on a smooth section transverse to the flow at $x^*(0) = 0$. Prove or disprove:

(i) $\mathcal{P}$ can undergo a period doubling bifurcation for $N \leq 2$.

(ii) $\mathcal{P}$ can undergo a torus bifurcation for $N \geq 3$.

Sketch the dynamics near a periodic orbit as it undergoes period doubling bifurcation.

For a vector space W with subspaces X, Y, let the direct sum $W = X \oplus Y$ denote that for every $w \in W$ there exist unique $x \in X$, $y \in Y$ such that $w = x + y$.
Let U, V be Banach spaces, $L : U \to V$ linear. We call L closed if graph(L) is closed, as a linear subspace of $U \times V$.

Closed Graph Theorem:[Theorem IV.4.5, D. Werner, *Funktionalanalysis*, Springer (2011)] If L as above is closed, then L is bounded (and therefore continuous).

Problem 14: Assume $W = X \oplus Y$ with W Banach and X closed in W (hence Banach). Let $Pw := x$ denote the linear projection of W onto X. Show that P is bounded and Y is closed, hence Banach.

Problem 15: Assume U, V are Banach, and $L : U \to V$ is a linear, closed bijection. Show that L and L^{-1} are bounded, i.e. L is a bounded linear isomorphism.

Problem 16: Assume U, V Banach, $L : U \to V$ is a bounded linear operator. Which of the following are true? Which of them are false? Justify your answers.

(i) The set of Fredholm operators $U \to V$ form a vector space.

(ii) If L is compact and Fredholm, then $\dim U < \infty$ and $\dim V < \infty$.

(iii) If $\dim U < \infty$ and $\dim V < \infty$, then L is Fredholm.

(iv) If $||L||$ is small, then L is Fredholm.

(v) Id_U is a compact operator.