Homework Assignments

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 17: Consider the Volterra equation

$$(1) \quad (Lx)(t) := x(t) - \int_0^t k(t,s)x(s)ds = f(t),$$

for a given integral kernel $k \in C^0([0,1]^2, \mathbb{R})$ and a forcing term

$$f \in C_0^0 := \{g \in C^0([0,1], \mathbb{R}) \mid g(0) = 0\}.$$

Show that $L : C_0^0 \to C_0^0$ is a Fredholm operator and indicate its Fredholm index.

[Extra credit]: Show that L has a trivial kernel and conclude that the integral equation (1) has a unique solution $x \in C_0^0$.

Problem 18: A group representation $\rho : \Gamma \to \mathrm{GL}(X)$ is called *faithful* if the homomorphism ρ is injective. Let ρ be a faithful representation of a finite group Γ on $\mathbb{C}$. Show that Γ is the cyclic group $\mathbb{Z}_N$ for some $N \in \mathbb{N}$.

Does the same claim hold for a faithful representation of a finite group on $\mathbb{R}^2$?

Problem 19: Let ρ be a representation of a topological group Γ with neutral element id on a Banach space X. Prove or disprove:

(i) ρ is strongly continuous if $\lim_{\gamma_n \to \mathrm{id}} \rho(\gamma_n) = \rho(\mathrm{id})$, in the operator norm of X.

(ii) ρ is strongly continuous *only if* $\lim_{\gamma_n \to \mathrm{id}} \rho(\gamma_n) = \rho(\mathrm{id})$, in the operator norm of X.

Problem 20: Let ρ be a representation of a group Γ on a Banach space X, and let K be a subgroup of Γ. For $x \in X$ let Γ_x denote the isotropy of x, and X_K the space of K-fixed vectors. For any $\gamma \in \Gamma$ show

(i) $x \in X_K$ if, and only if, $\Gamma_x \geq K$;

(ii) $\Gamma_{\gamma x} = \gamma \Gamma_x \gamma^{-1}$;

(iii) $\gamma X_K = X_{\gamma K \gamma^{-1}}$;

(iv) Γ_x is a normal subgroup of Γ if, and only if, the isotropy of any y in the Γ-orbit of x is Γ_x .