Homework Assignments

Dynamical Systems III

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 25: Let ρ be a representation of a group Γ in $\mathbb{R}^N$. Consider a Γ-equivariant C^k-vector field, $k \geq 1$,

$$\dot{x} = f(x), \; x \in \mathbb{R}^N.$$

Assume that $f(0) = 0$ and $A = f'(0)$ possesses a nontrivial center eigenspace.
Prove or disprove:

(i) If A and f are close enough in $BC^1(\mathbb{R}^N, \mathbb{R}^N)$, the global center manifold W^c is Γ-invariant.

(ii) Any local center manifold $W^c_{\text{loc}}(0)$ is Γ-invariant.

Problem 26: Let ρ be a representation of a group Γ on V.Let K be a subgroup of Γ such that $\dim V_K = 1$ and let Γ_u denote the isotropy of any $0 \neq u \in V_K$.

(i) Assume $\Gamma_u \neq \Gamma$. Show that Γ_u is a maximal isotropy subgroup of Γ on V.

(ii) Assume ρ is irreducible and $V_K \neq V$. Show that $V_\Gamma = \{0\}$ and hence $\Gamma_u \neq \Gamma$.

Problem 27: Let $\Gamma = \mathbb{R} \times S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$, act via ρ on the functions on a cylinder $u \in C^2(\mathbb{R} \times S^1, \mathbb{R})$ by translation, i.e.

$$\rho(\tau, \xi)u(t, x) = u(t + \tau, x + \xi).$$

Fix a real number $c \neq 0$ and consider the subgroup $K_c := \{(\tau, c\tau) \mid \tau \in \mathbb{R}\} \leq \Gamma$. We say that a K_c-fixed function is a *traveling wave* with *wavespeed* c. Show that any traveling wave with wavespeed c on the cylinder satisfies

$$\partial_t u + c\partial_x u = 0.$$

Show that the Sine-Gordon equation on the cylinder

$$\partial_{tx} u = \sin(u), \ (t, x) \in \mathbb{R} \times S^1,$$

is Γ-equivariant. Discuss how traveling waves of the Sine-Gordon equation relate to solutions of the equation

$$v'' + \lambda \sin(v) = 0, \ v \in C^2(\mathbb{R} \times S^1, \mathbb{R}), \ \lambda \in \mathbb{R}.$$

Problem 28: Consider the equation

$$F(\lambda, v) := v'' + \lambda \sin v = 0$$

on the space of 1-periodic functions, $F : \mathbb{R} \times C^2(S^1, \mathbb{R}) \to C^0(S^1, \mathbb{R})$, $S^1 = \mathbb{R}/\mathbb{Z}$.

(i) Determine the bifurcation points of F, i.e. the values of λ at which $D_v F(\lambda, 0)$ has a non-trivial kernel. Is the kernel one-dimensional?

(ii) Consider the involution $\kappa : C^0(S^1, \mathbb{R}) \to C^0(S^1, \mathbb{R})$ given by

$$(\kappa v)(x) = v(-x),$$

and let $K := \langle \kappa \rangle$ denote the group $\mathbb{Z}_2$ generated by κ. Show that the fixed-point subspace $C^2(S^1, \mathbb{R})_K$ can be identified with the subspace of functions in $C^2([0, 1/2], \mathbb{R})$ with Neumann ($\{v'(0) = v'(1/2) = 0\}$) boundary conditions.

[Extra credit]: Discuss the bifurcation diagram for the traveling wave solutions of the Sine-Gordon equation above.