Homework Assignments

Dynamical Systems III

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Problem 29: Consider $\Gamma \coloneqq (\mathbb{R}/\mathbb{Z}, +)$ and let $\sigma : \Gamma \times \mathbb{R}^N \to \mathbb{R}^N$ be a C^1-action of Γ on $\mathbb{R}^N$, i.e. σ is a C^1-map and

(i) $\sigma(0, x) = x$,

(ii) $\sigma(\gamma_1 + \gamma_2, x) = \sigma(\gamma_1, \sigma(\gamma_2, x))$, for all $\gamma_1, \gamma_2 \in \Gamma$ and $x \in \mathbb{R}^N$.

Assume that σ fixes 0, i.e. $\sigma(\gamma, 0) = 0$ for all $\gamma \in \Gamma$. Show that:

(a) $\rho(\gamma) \coloneqq D_x\sigma(\gamma, 0)$ is a representation of Γ on $\mathbb{R}^N$.

(b) The map

$$\Psi(x) \coloneqq \int_0^1 \rho(-\alpha)\sigma(\alpha, x) d\alpha$$

defines a C^1-diffeomorphism for x near 0.

(c) Ψ conjugates σ to ρ locally, i.e.

$$\rho(\gamma)\Psi(x) = \Psi(\sigma(\gamma, x)), \text{ for all } \gamma \in \Gamma \text{ and } x \text{ near } 0.$$

Extra credit: With Haar measure $d\mu(\gamma)$ on Γ and

$$\Psi(x) \coloneqq \int_\Gamma \rho(\gamma^{-1})\sigma(\gamma, x) d\mu(\gamma),$$

this extends to C^1-actions σ of arbitrary compact Lie groups Γ, verbatim.

Problem 30: Consider the ODE in $\mathbb{C}^2 \cong \mathbb{R}^4$ given by

$$\begin{aligned} \dot{z}_1 &= (\lambda + i)z_1 + \alpha|z_1|^2 z_2 + \beta|z_2|^2 z_1, \\ \dot{z}_2 &= (\lambda + i)z_2 + \alpha|z_2|^2 z_1 + \beta|z_1|^2 z_2. \end{aligned}$$

Does the ODE satisfy the assumptions of the Hopf bifurcation theorem at parameter $\lambda = 0$? Does it satisfy the conclusions?

Find a proper invariant subspace where the assumptions of the Hopf bifurcation theorem are met.

Problem 31: Consider the Hamiltonian system

$$\dot{x} = J\nabla H(x), \qquad J = \begin{pmatrix} 0 & -\mathrm{Id}_N \\ \mathrm{Id}_N & 0 \end{pmatrix}, \qquad H \in C^2(\mathbb{R}^{2N}, \mathbb{R}).$$

Let the origin $x = 0$ be an isolated equilibrium: $\nabla H(0) = 0$. Assume that the linearization $JD^2H(0)$ possesses a pair of algebraically simple eigenvalues $\pm\mathrm{i}$, and that all other eigenvalues have non-vanishing real parts.

Prove that in a neighborhood of the origin there is an invariant 2-dimensional manifold filled with periodic orbits.

Hint: Consider the system $\dot{x} = (\lambda \mathrm{Id}_{2N} + J)\nabla H(x)$ with a real parameter λ and discuss the Hopf bifurcation which occurs at $\lambda = 0$.

Problem 32: For each of the following cases, find an example of a vector field which is:

(i) Hamiltonian, but not reversible.

(ii) Reversible, but not Hamiltonian.

(iii) Both Hamiltonian and reversible.

(iv) Neither Hamiltonian nor reversible.