Homework Assignments

Dynamical Systems III

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`http://dynamics.mi.fu-berlin.de/lectures/`

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Problem 33: [J. Yorke *Periods of periodic solutions and the Lipschitz constant*](1969) Let $x(t) \in \mathbb{R}^N$ be a nonstationary periodic solution of $\dot{x} = f(x)$ with minimal period $p > 0$. Assume $f \in C^1$ and let $c := \sup_x |f'(x)|_2$, with the Euclidean norm $|\cdot|_2$.

(i) Rescale the minimal period p of $x(t)$ to the minimal period 2π of $\tilde{x}(t) := x(\frac{p}{2\pi}t)$.

(ii) Derive an ODE for $\eta := \frac{d}{dt}\tilde{x}(t)$, which involves $f'(\tilde{x}(t))$.

(iii) Show that $\eta(t)$ possesses mean value zero.

(iv) We know the Poincaré inequality (Fourier expansion!), i.e.

$$(*) \qquad ||\eta||_2^2 \leq ||\dot{\eta}||_2^2$$

holds, for the Euclidean L^2-norms $||\eta||_2^2 := \int_0^{2\pi} |\eta(t)|_2^2 dt$ and any 2π-periodic function $\eta \in C^1(\mathbb{R}, \mathbb{R}^N)$ of mean value $\frac{1}{2\pi}\int_0^{2\pi} \eta(t)dt = 0$.
Use $(*)$ to conclude the following lower bound on minimal periods p:

$$p \geq 2\pi/c.$$

(v) Is the lower bound $2\pi/c$ for p sharp?

Problem 34: Let $x_n(t)$ be a sequence of periodic solutions of $\dot{x} = f(x)$, $f \in C^1$, with minimal period p_n. Let $||x_n(\cdot)||$ denote the sup-norm and assume $||x_n(\cdot)|| \to 0$ and $p_n \to p \in (0, \infty)$, for $n \to \infty$.

(i) Show $f(0) = 0$.

(ii) Show that $y_n(\cdot) := x_n(\cdot)/||x_n(\cdot)||$ possesses a convergent subsequence $y_n \to y$, in C^0, and derive an ODE for $y(t)$.

(iii) Conclude that $A := f'(0)$ possesses a purely imaginary eigenvalue $i\omega$.

Free extra: Show $\omega \neq 0$.

Problem 35: Consider an R-reversible C^1 vector field

$$\dot{x} = f(x) \in \mathbb{R}^N, \qquad f(Rx) = -Rf(x), \qquad f(0) = 0, \qquad A := f'(0),$$

with R linear, $R^2 = \text{id}$, $\text{Fix}(R) := \{x \in \mathbb{R}^N \mid Rx = x\}$.

(i) Prove that $\det(A) \neq 0$ implies N is even.

(ii) For N odd and

$$\dim \text{Fix}(R) = (N+1)/2.$$

Show that if $\dim \text{Ker} A < 2$, there exists a local curve of equilibria in $\text{Fix}(R)$, locally near 0.

Problem 36: Consider a smooth vector field

$$\dot{x} = f(\lambda, x), \qquad x \in \mathbb{R}^2, \qquad \lambda \in \mathbb{R}, \qquad f(\lambda, 0) \equiv 0, \qquad D_x f(\lambda, 0) = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix},$$

with flow Φ_t^f. Characterize the parameter values λ, T at which the fixed point $x = 0$ of the iteration

$$x_{n+1} = \Phi_T^f(x_n),$$

undergoes subharmonic bifurcation. Discuss and interpret how the subharmonic bifurcation for the iteration relates to the Hopf bifurcation happening at parameter $\lambda = 0$ for the original ODE.