Übungen zur Vorlesung

Analysis II

Sommersemester 2022

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`http://dynamics.mi.fu-berlin.de/lectures/`

Due date: Wednesday, 04.05.2022, 17:00.

Solutions in German or English, please.

Problem 1: Consider the Banach space $(\mathbb{R}, \|\cdot\|)$, where $\|\cdot\|$ denotes any norm on $\mathbb{R}$. Where is the *norm map* $\|\cdot\| : \mathbb{R} \to [0, \infty)$ continuous (stetig)? We say that $\|\cdot\|$ is differentiable (differenzierbar) at $x \in \mathbb{R}$ if there exists $A \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{\|x + h\| - \|x\| - Ah}{\|h\|} = 0.$$

Is $\|\cdot\|$ differentiable at 0?

Problem 2: Let X and Z be Banach spaces and consider the space $BL(X, Z)$ of all linear operators $L : X \to Z$ with finite operator norm

$$\|L\| := \sup_{\|x\|_X = 1} \|Lx\|_Z.$$

Show that:

(i) The operator norm satisfies

$$\|L\| = \sup_{x \neq 0} \frac{\|Lx\|_Z}{\|x\|_X}.$$

(ii) $BL(X, Z)$ equipped with the operator norm is a Banach space.

Problem 3: Let P_N denote the real vector space of real polynomials of degree (Grad) $\leq N$ with domain $[0,1]$, equipped with the sup-norm $\|p\|_{C^0} := \sup_{x\in[0,1]} |p(x)|$. Prove or disprove:

(i) P_N is a Banach space.

(ii) The derivative (Ableitung) $p \mapsto p'$ is a surjective linear operator mapping P_N back to itself.

(iii) The derivative $p \mapsto p'$ is a bounded linear operator on P_N.

Extra credit: Consider the union P of all P_N, $N \in \mathbb{N}$, instead.

Problem 4: Consider $C^0([0,1],\mathbb{R})$ and $C^1([0,1],\mathbb{R})$. Let us denote the sup-norm by $\|f\|_{C^0} := \sup_{x\in[0,1]} |f(x)|$ and the C^1-norm by $\|f\|_{C^1} := \|f(x)\|_{C^0} + \|f'(x)\|_{C^0}$. For the *derivative map* $' : f \mapsto f'$, prove or disprove:

(i) $' : (C^1([0,1],\mathbb{R}), \|\cdot\|_{C^1}) \to (C^0([0,1],\mathbb{R}), \|\cdot\|_{C^0})$ is a bounded linear operator.

(ii) $' : (C^1([0,1],\mathbb{R}), \|\cdot\|_{C^0}) \to (C^0([0,1],\mathbb{R}), \|\cdot\|_{C^0})$ is a bounded linear operator.

In case the derivative map is bounded, determine the operator norm $\|\cdot'\|$.