Übungen zur Vorlesung

Analysis II

Sommersemester 2022

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http://dynamics.mi.fu-berlin.de/lectures/

Due date: Wednesday, 25.05.2022, 17:00.

Solutions in German or English, please.

Problem 13: Determine the partial fraction decompositions (Partialbruchzerlegungen) of two of the following functions $f : (0, \infty] \to \mathbb{R}$, and determine their primitives $F(x) := \int_1^x f(t)\,\mathrm{d}t$:

(i) $f(t) = \dfrac{1}{(t+1)(t^2-2t+5)}$;

(ii) $f(t) = \dfrac{1}{t^3+8}$;

(iii) $f(t) = \dfrac{1}{t(t+2)^3}$.

Problem 14: Find a sequence of integrable functions $f_n : [0,1] \to \mathbb{R}$, that converge pointwise towards a function f such that

(i) f is not integrable;

(ii) f is integrable and

$$\int_0^1 f(t)\,\mathrm{d}t \;\neq\; \lim_{n\to\infty} \int_0^1 f_n(t)\,\mathrm{d}t.$$

Consider either the regulated or the Riemann integral.
(Free extra points: Consider both!)

Problem 15: Show that the function

$$f(t) := \begin{cases} \frac{1}{q}, & \text{if } t = \frac{p}{q} \text{ with } p, q \in \mathbb{N}, \text{ such that } \gcd(p,q) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

is Riemann integrable for $t \in [0,1]$ and determine the value of the integral.

gcd stands for greatest common divisor (größter gemeinsamer Teiler).

Problem 16: Consider the unit interval $C_0 := [0,1]$ and define C_n to be C_{n-1} after removing the open middle third from each connected component, i.e., $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so on.

The *Cantor set* (Cantor-Menge) C is defined as the countable intersection

$$C := \bigcap_{n \geq 0} C_n.$$

Show that C is an uncountable (überabzählbar) Lebesgue null set (Nullmenge).

Problem E1(Extra credit): A student defines a set $A \subset [0,1]$ to be *S-null*, if for every $\varepsilon > 0$ there exists a cover $A \subset \bigcup_k I_k$ by open and **disjoint** intervals I_k such that

$$\sum_k |I_k| < \varepsilon.$$

Prove or disprove for A:

(i) S-null implies dense;

(ii) S-null implies not dense;

(iii) S-null implies Lebesgue null;

(iv) Lebesgue null implies S-null.