Homework

V19028: Dynamical Systems II

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Problem 5: [Billiard] Consider an ideal square billiard table Q (without holes, i.e. three-cushion billiard rather than snooker) and a pointsize (no side-spin effects) frictionless ball moving on it. Thus, the billiard ball moves at constant speed and on straight lines. At the boundary of the table it is reflected such that the angle of incidence is equal to the angle of reflection.

Let the initial slope of the trajectory of the ball with respect to the boundary be irrational. Let M be an open subset of Q, and m the ratio of the area of M and the area of Q. Let $\mu(T)$ denote the total length of all time intervals (τ, t), $0 \leq \tau < t \leq T$, in which the position of the ball is in M. Prove

$$\lim_{T\to\infty} \frac{\mu(T)}{T} = m.$$

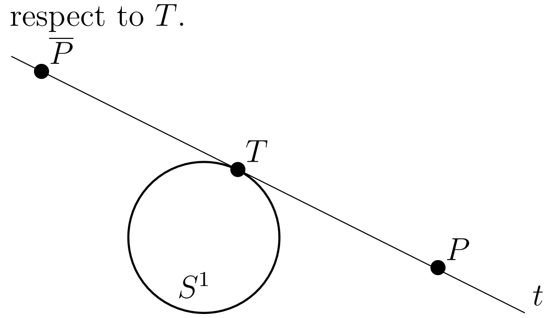
Free extra: What happens for rectangular billiard tables? What happens in 3 space dimensions, i.e. in a cube?

Problem 6: [Outer Billiard] Starting with the unit circle $S^1 = \partial B_1(0)$ in the plane, we define the *outer billiard* as the map on its exterior

$$F : \mathbb{R}^2 \backslash B_1(0) := \left\{ P = (x,y) \in \mathbb{R}^2 \,\middle|\, \|P\|_2 = \sqrt{x^2+y^2} > 1 \right\} \longrightarrow \mathbb{R}^2 \backslash B_1(0), \qquad P \longmapsto \overline{P}$$

by the following construction.

Let t be the line through P tangent to the unit circle S^1 on its right side as seen from P. Let T be the intersection point of t with the unit circle S^1. Then $\overline{P}$ is defined as the mirror image of P with respect to T.



(i) Prove that F maps each circle centered at the origin onto itself.

(ii) Prove that F is a homeomorphism on each of these circles.

(iii) Determine the rotation number of F.

Free extra Discuss the outer billiard with respect to an ellipse.

Problem 7: Consider the vector field

$$\dot{y} = f(y), \qquad y \in S^1, \quad f(y) > 0$$

with corresponding flow φ_t. Determine a formula for the rotation number $\varrho(\varphi_T)$ of the time-T-map, for example by separation of variables. Does a devil's staircase arise?

Problem 8: Consider the map $A : \mathbb{R} \to \mathbb{R}$,

$$A(y) \;=\; \begin{cases} 2y, & 0 \leq y < 1 \\ A(y-1)+1, & 1 \leq y \\ A(y+1)-1, & y < 0 \end{cases}$$

Thus $A(y+1) = A(y) + 1$ for all y and A defines a map $\tilde{A} : S^1 \to S^1 = \mathbb{R}/\mathbb{Z}$. However A and $\tilde{A}$ are not homeomorphisms. Nonetheless, try to define a "rotation number" $\varrho(y_0)$ for initial conditions y_0. Does $\varrho(y_0)$ depend on y_0?