Homework

V19028: Dynamical Systems II

Bernold Fiedler, Stefan Liebscher

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Problem 13: Consider a diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ with $\Phi(0) = 0$. Let

$$\begin{aligned} W^s &= \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \Phi^n(x) = 0\} \\ W^u &= \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \Phi^{-n}(x) = 0\} \end{aligned}$$

denote the stable and the unstable set of the origin. Find — if possible — an example and a counterexample for each of the following cases:

(i) W^s is an embedded submanifold.

(ii) W^s is closed.

(iii) $W^s \cap W^u$ consists of exactly two distinct points.

Free extra: Find an example such that W^s is not even a manifold.

Reminder: W is a manifold of dimension M if it is locally homeomorph to $\mathbb{R}^M$. W is an *embedded submanifold* of dimension M in $\mathbb{R}^N$ if for all $x \in W$ there exists a Ball $x \in B_\varepsilon(x) \subset \mathbb{R}^N$ and a homeomorphism $h : B_\varepsilon(x) \to B_1(0)$ such that $h(W \cap B_\varepsilon(x)) = (0 \times \mathbb{R}^M) \cap B_1(0)$.

Problem 14: Consider the pendulum

$$\ddot{\varphi} + \sin\varphi \;=\; 0.$$

Let

$$W^{\mathrm{s}}_{\mathrm{loc}} \;=\; \{(\varphi, \dot{\varphi}) = (\varphi, h(\varphi - \pi))\ ;\ \pi - \varepsilon < \varphi < \pi + \varepsilon\}$$

be the local stable manifold at the equilibrium $\varphi = \pi$, $\dot{\varphi} = 0$. Determine the expansion

$$h(\psi) \;=\; \sum_{k=0}^{N} h_k \psi^k + \mathcal{O}(\psi^{N+1})$$

up to order $N = 3$.

Hint: Use the invariance of W^{S}.

Free extra: Consider the damped pendulum.

Problem 15: Consider the damped pendulum

$$\ddot{\varphi} + \alpha\dot{\varphi} + \sin\varphi \;=\; 0,$$

with $\alpha > 0$ and $\varphi \in \mathbb{R}$.

(i) Sketch the stable manifold of the equilibrium $\varphi = \pi$, $\dot{\varphi} = 0$ for $\alpha > 0$.

(ii) How do trajectories above and below the stable manifold differ?

Problem 16: Consider the map

$$f \,:\, S^1 \to S^1 \;=\; \mathbb{R}/\mathbb{Z}, \qquad f(y) \;:=\; 2y \pmod 1,$$

see also Problem 8. Prove that f contains periodic orbits of every period. Are there orbits which are dense on the circle S^1?

Free extra: What can you say about $f(y) = gy \pmod 1$ for integer factors $g \geq 2$?