Homework

V19028: Dynamical Systems II

Bernold Fiedler, Stefan Liebscher

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Problem 17: Let

$$\Sigma_N \;=\; \Big\{ s = (s_j)_{j\in\mathbb{Z}} \mid s_j \in \{0,\ldots,N-1\} \Big\}$$

be the set of sequences on N symbols, with the metric

$$\operatorname{dist}(s,s') \;:=\; \sum_{j\in\mathbb{Z}} (2N)^{-|j|}|s_j - s'_j|.$$

Consider the shift

$$\sigma \;:\; \Sigma_N \to \Sigma_N, \quad (s_j)_{j\in\mathbb{Z}} \mapsto (s_{j+1})_{j\in\mathbb{Z}}.$$

What are the fixed points of σ ? Determine the stable and unstable sets and thus all homoclinic and heteroclinic orbits of these fixed points.

Problem 18: Let $\Phi : M \to M$ be a continuous map on a metric space M. We call a sequence $(\xi_k)_{k\in\mathbb{N}}$ a δ-*pseudo orbit*, if the estimate

$$\operatorname{dist}(\Phi(\xi_k),\xi_{k+1}) \;<\; \delta.$$

holds for all $k \in \mathbb{N}$. We call a Φ-orbit $(x_k)_{k\in\mathbb{N}} = (\Phi^k(x_0))_{k\in\mathbb{N}}$ in M an ε-*shadow* of the pseudo orbit $(\xi_k)_{k\in\mathbb{N}}$ if the estimate

$$\operatorname{dist}(x_k,\xi_k) \;<\; \varepsilon$$

holds for all $k \in \mathbb{N}$. We say that the pair (M,Φ) has the *shadow property*, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo orbit has an ε-shadow.

(i) Prove: the shift on two symbols has the shadow property.

(ii) Give an interpretation of the shadow property from a numerical view point.

Free extra: Is the shadow unique?

Problem 19: Calculate all fixed points of the bouncing-ball map f:

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma\cos(\Phi_j + v_j) \end{aligned}$$

with $\Phi_j \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and $v_j \in \mathbb{R}$, for $0 < \alpha < 1$ and $0 < \gamma$. How many fixed points does f have for given α, γ? Determine the type (stable, unstable, non-hyperbolic etc.) of the fixed points. Sketch the dependence of the fixed points on γ, for $\alpha = \frac{1}{2}$. What happens for $\alpha \to 1$?

Problem 20: Consider again the bouncing-ball map $f_{\alpha,\gamma}$ on $S^1 \times \mathbb{R}$:

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma \cos(\Phi_j + v_j), \end{aligned}$$

with $0 < \alpha < 1$ and $0 < \gamma$. Define the domain

$$D := \left\{ (\Phi, v) \in S^1 \times \mathbb{R} \ : \ |v| \leq \frac{\gamma}{1-\alpha} + \varepsilon \right\}$$

for some $\varepsilon > 0$. Prove:

(i) D is positively invariant, i.e. $f_{\alpha,\gamma}(D) \subseteq D$.

(ii) D is absorbing, i.e. for all (Φ_0, v_0) there exists $n_0 \in \mathbb{N}$ such that $(\Phi_n, v_n) \in D$ for all $n \geq n_0$.

(iii) The *global attractor*, defined by

$$\mathcal{A}_{\alpha,\gamma} := \bigcap_{n=0}^{\infty} f^n_{\alpha,\gamma}(D),$$

is compact and invariant under $f_{\alpha,\gamma}$ as well as $f^{-1}_{\alpha,\gamma}$. Furthermore, $\mathcal{A}_{\alpha,\gamma}$ is the *maximal* compact and invariant set.

(iv) $\mathcal{A}_{\alpha,\gamma}$ is indeed attracting, i.e. for all (Φ_0, v_0)

$$\lim_{n \to \infty} \operatorname{dist}\Big((\Phi_n, v_n), \mathcal{A}_{\alpha,\gamma} \Big) = 0.$$

(v) $\mathcal{A}_{\alpha,\gamma}$ contains the closure of the set of all periodic points of $f_{\alpha,\gamma}$.